Representations of conformal Galilei algebra with integer spin and an application

N. Aizawa

Osaka Prefecture University, Japan

Joint work with R. Chandrashekar, J. Segar

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Today's topic

Conformal Galilei algebra

- non-relativistic version of conformal algebra $so(n, 2)$
- class of non-semisimple Lie algebras

  each member $\cdots$ labelled by 2-parameters d, ℓ

  $d = 1,\ 2,\ 3,\ \ldots$ (space dimension)

  $\ell = \frac{1}{2},\ 1,\ \frac{3}{2},\ 2, \frac{5}{2},\ \ldots$
- central extensions if $(d, \ell) = (\forall, \text{half-int})$ or $(2, \text{int})$

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- central extensions if $(d,\ell) = (\forall, \text{half-int})$ or $(2, \text{int})$

In this talk

$d = 1$ and integer ℓ $\Rightarrow$ no central extensions

some representation theory and its application to differental equations

Motivation

Relevance of conformal algebra in non-relativistic physics

- non-relativistic AdS/CFT correspondence
  Son 2008, Balasubramanian, McGreevy 2008, Alishahiha, Davody, Vahedi 2009, Bagchi, Gopakumar 2009, Martelli, Tachikawa 2010, NA, Dobrev 2010
- quantum many-body systems Galajinsky 2008
- classical mechanics/fileds Lukierski, Stichel, Zakrzewski 2006 2007 Galajinsky, Masterov 2013, Andrzejewski *et al* 2012, 2013
- Galilean electrodynamics Negro, del Olmo, Rodriguez-Marco 1997
- geometry of Newtonian mechanics Duval, Horváthy 2009, 2011
- gravity Hotta *et al* 2010, Bagchi *et al* 2010, 2012, Barnich *et al* 2012
- fluid mechanics O'Raifeartaigh, Sreedhar 2001, Zhang, Horváthy 2010
- anisotropic statistical systems Henkel 1994, 1997, 2002 *etc*

Motivation

Representation theories of conformal Galilei algebra

$\cdots$ mathematical/physical importance

Previous works on *irreducible lowest/highest* weight modules

$d \backslash \ell$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\cdots$
1							
2							
3							
4							
$\vdots$							

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$d \backslash \ell$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\cdots$
1	●						
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3	●						
4							
$\vdots$							

● Dobrev, Doebner, Mrugalla 1997, Mrugalla 1997

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$d \backslash \ell$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\cdots$
1	● (blue)						
2	● (blue)	● (red)					
3	● (blue)						
4							
$\vdots$							

● (blue) Dobrev, Doebner, Mrugalla 1997, Mrugalla 1997 ● (red) NA, Isaac 2011

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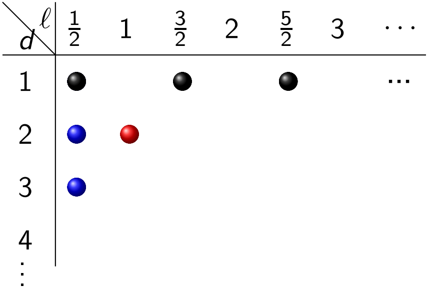

Dobrev, Doebner, Mrugalla 1997, Mrugalla 1997

NA, Isaac 2011

NA, Isaac, Kimura 2012

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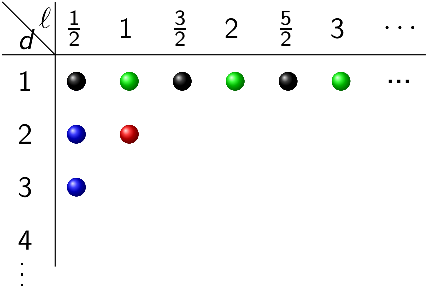

Dobrev, Doebner, Mrugalla 1997, Mrugalla 1997 — NA, Isaac 2011

NA, Isaac, Kimura 2012 — present work

Lü, Mazorchuk, Zhao, J. Pure Appl. Algebra **218** (2014) 1885 $\Rightarrow$ $d = 1$ and any ℓ

Outline

- $d = 1$ Conformal Galilei algebras $\mathfrak{g}_\ell$ with integer ℓ
  - definition and properties
- Lowest weight Verma modules over $\mathfrak{g}_\ell$
  - definition
  - Kac determinant, singular vectors $\Rightarrow$ reducibility of Verma modules
- Application : differential equations invariant under the group generated by $\mathfrak{g}_\ell$
- Irreducible lowest weight modules of $\mathfrak{g}_\ell$
- Summary

$d = 1$ Conformal Galilei algebras $\mathfrak{g}_\ell$

Negro, del Olmo, Rodriguez-Marco 1997

Transformations in $(1+1)$D spacetime (t, x)

1. Non-relativistic kinematical algebra (Galilei algebra)

$$H = \frac{\partial}{\partial t}, \qquad P_0 = \frac{\partial}{\partial x}, \qquad P_1 = -t\frac{\partial}{\partial x},$$

2. Special conformal transformation + dilatation

$$C : t \to \frac{t}{1-at}, \quad x \to \frac{x}{(1-at)^{2\ell}} \quad D : t \to at, \quad x \to a^{\ell} x$$

$$\ell = 1, 2, 3, \ldots$$

3. Additional Abelian generators

$$P_m = (-t)^n \frac{\partial}{\partial x}, \qquad n = 2, 3, \cdots, 2\ell$$

$d = 1$ Conformal Galilei algebras $\mathfrak{g}_\ell$ Negro, del Olmo, Rodriguez-Marco 1997

Generators

$$D, \ H, \ C, \ P_n, \quad (n = 0, 1, \ldots, 2\ell)$$

Non-vanishing commutators

$$[D, H] = H, \qquad [D, C] = -C, \qquad [C, H] = 2D,$$
$$[H, P_n] = -nP_{n-1}, \qquad [D, P_n] = (\ell - n)P_n, \qquad [C, P_n] = (2\ell - n)P_{n+1}.$$

Structure

- $\{ \ H, D, C \ \} \ \Rightarrow \ so(1, 2) \simeq sl(2, \mathbb{R})$ subalgebra
- $\{ \ P_n \ \}_{n=0,1,\ldots,2\ell} \ \Rightarrow$ abelian ideal of $\mathfrak{g}_\ell$

  spin ℓ rep of $sl(2, \mathbb{R})$ subalgebra

$$so(1, 2) \ni \{ \ P_n \ \}$$

Lowest weight representations of $\mathfrak{g}_\ell$

"Triangular decomposition" $\mathfrak{g}_\ell = \mathfrak{g}_\ell^+ \oplus \mathfrak{g}_\ell^0 \oplus \mathfrak{g}_\ell^-$

$$\mathfrak{g}_\ell^+ = \langle\ H,\ P_0,\ P_1,\ \dots,\ P_{\ell-1}\ \rangle$$

$$\mathfrak{g}_\ell^0 = \langle\ D,\ P_\ell\ \rangle \qquad \Rightarrow \quad [\mathfrak{g}_\ell^0, \mathfrak{g}_\ell^\pm] \subseteq \mathfrak{g}_\ell^\pm$$

$$\mathfrak{g}_\ell^- = \langle\ C,\ P_{\ell+1},\ P_{\ell+2},\ \dots,\ P_{2\ell}\ \rangle$$

Lowest weight vector $|\delta, p\rangle$

$$D\,|\delta,p\rangle = \delta\,|\delta,p\rangle\,, \qquad P_\ell\,|\delta,p\rangle = p\,|\delta,p\rangle\,, \qquad X\,|\delta,p\rangle = 0, \quad {}^\forall X \in \mathfrak{g}_\ell^-$$

Verma module $V_\ell^{\delta,p}$ $\cdots$ representation space of $\mathfrak{g}_\ell$

$$V_\ell^{\delta,p} = U(\mathfrak{g}_\ell^+)\,|\delta,p\rangle = \left\{\ H^k P_{\ell-1}^{m_1} P_{\ell-2}^{m_2} \cdots P_0^{m_\ell}\,|\delta,p\rangle \ \mid\ k, m_i \in \mathbb{Z}_{\geq 0}\ \right\}$$

Lowest weight representations of $\mathfrak{g}_\ell$

"Triangular decomposition" $\mathfrak{g}_\ell = \mathfrak{g}_\ell^+ \oplus \mathfrak{g}_\ell^0 \oplus \mathfrak{g}_\ell^-$

$$\mathfrak{g}_\ell^+ = \langle\ H,\ P_0,\ P_1,\ \dots,\ P_{\ell-1}\ \rangle \leftarrow \text{creation}$$

$$\mathfrak{g}_\ell^0 = \langle\ D,\ P_\ell\ \rangle \leftarrow \text{diagonal} \qquad \Rightarrow \quad [\mathfrak{g}_\ell^0, \mathfrak{g}_\ell^\pm] \subseteq \mathfrak{g}_\ell^\pm$$

$$\mathfrak{g}_\ell^- = \langle\ C,\ P_{\ell+1},\ P_{\ell+2},\ \dots,\ P_{2\ell}\ \rangle \leftarrow \text{annihilation}$$

Lowest weight vector $|\delta, p\rangle$ $\leftarrow$ vacuum

$$D\,|\delta, p\rangle = \delta\,|\delta, p\rangle\,, \qquad P_\ell\,|\delta, p\rangle = p\,|\delta, p\rangle\,, \qquad X\,|\delta, p\rangle = 0, \quad {}^\forall X \in \mathfrak{g}_\ell^-$$

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$\leftarrow$ creation ops on vacuum

Lowest weight representations of $\mathfrak{g}_\ell$

Why Verma modules ?

1. $V_\ell^{\delta,p}$ is the largest lowest weight module: eg. Dixmier 1977

other lowest weight modules are obtained from the Verma modules

2. an application Dobrev 1988

$V_\ell^{\delta,p}$ is reducible $\Rightarrow$ invariant differential equations

invariance = kinematical symmetry under the group generated by $\mathfrak{g}_\ell$

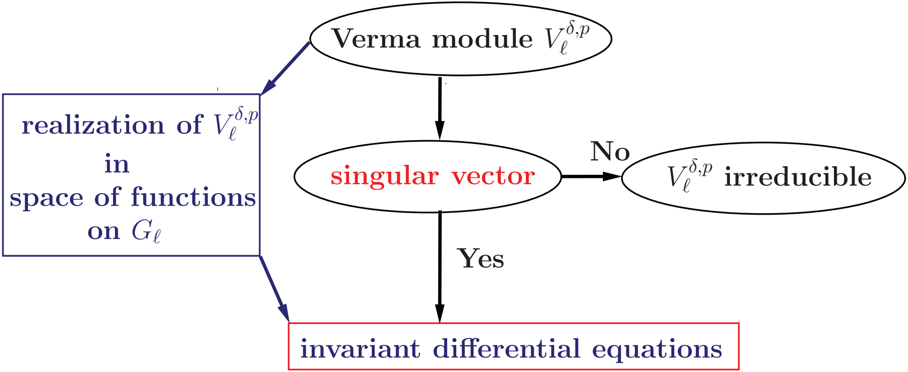
Eg. $\ell = 1/2$ *conformal Galilei algebra* $\Rightarrow$ *free Schrödinger equation*

Dobrev, Doebner, Mrugalla 1997, NA, Dobrev, Doebner 2002

NA, Dobrev, Doebner, Stoimenov 2008

Lowest weight representations of $\mathfrak{g}_\ell$

Reducibility, invariant equations, How ? $\Leftarrow$ singular vector



G_ℓ : group generated by $\mathfrak{g}_\ell$

Singular vector $|v_s\rangle$

definition

Another lowest weight vector in $V_\ell^{\delta,p}$, $|v_s\rangle \neq \mathbb{C}\,|\delta, p\rangle$

$$D\,|v_s\rangle = \delta'\,|v_s\rangle\,, \qquad P_\ell\,|v_s\rangle = p'\,|v_s\rangle\,, \qquad X\,|v_s\rangle = 0, \quad {}^\forall X \in \mathfrak{g}_\ell^-$$

- $|v_s\rangle$ exists $\Rightarrow$ exists invariant subspace $\mathcal{I} \subset V_\ell^{\delta,p}$ $\Rightarrow$ $V_\ell^{\delta,p}$ reducible

  $(\because)$ $\mathcal{I} = U(\mathfrak{g}_\ell^+)\,|v_s\rangle \subset V_\ell^{\delta,p} = U(\mathfrak{g}_\ell^+)\,|\delta, p\rangle$

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- $|v_s\rangle$: orthogonal to any vector in $V_\ell^{\delta,p}$

  $(\because)\ \ |u\rangle = X\,|\delta,p\rangle \in V_\ell^{\delta,p},\ \langle u|v_s\rangle = \langle \delta,p|\,X^\dagger\,|v_s\rangle = 0$

  X ↑ creation, $X^\dagger$ ↑ annihilation

Detection of singular vectors - Kac determinant -

Kac determinant eg. Kac, Raina 1987

- quantity defined in a *fixed level* subspace of $V_\ell^{\delta,p}$ (next slide)

$^\exists$singular vector at level N subspace $\Rightarrow$ level N Kac determinant $= 0$

i.e.,

Kac determinant $\neq 0$ for all N $\Rightarrow$ $V_\ell^{\delta,p}$ is irreducible

Kac determinant of $\mathfrak{g}_\ell$ - examples for small ℓ -

$\ell = 1$ algebra $\mathfrak{g}_1$

- basic data

$$\mathfrak{g}_1^+ = \langle\ H, P_0\ \rangle \ \stackrel{\dagger}{\longleftrightarrow}\ \mathfrak{g}_1^- = \langle\ C, P_2\ \rangle, \qquad \mathfrak{g}_1^0 = \langle\ D, P_1\ \rangle = (\mathfrak{g}_1^0)^\dagger$$

$$V_1^{\delta,p} = \{\ |k,m\rangle = H^k P_0^m\,|\delta,p\rangle\ \}, \quad D\,|\delta,p\rangle = \delta\,|\delta,p\rangle\,, \quad P_1\,|\delta,p\rangle = p\,|\delta,p\rangle$$

- graded structure of $V_1^{\delta,p}$: $D\,|k,m\rangle = (\delta + k + m)\,|k,m\rangle$

$N = k + m$	vectors
levl 0	$\|0,0\rangle$
levl 1	$\|1,0\rangle\,,\ \|0,1\rangle$
levl 2	$\|2,0\rangle\,,\ \|1,1\rangle\,,\ \|0,2\rangle$
$\vdots$	$\dots$

Kac determinant of $\mathfrak{g}_\ell$ - examples for small ℓ -

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$\vdots$	$\dots$

singular vector
$\Downarrow$
eigenvector of D
$\Downarrow$
exist in level N subspace

Detection of singular vectors - Kac determinant -

- level N Kac determinant

$$\Delta_N^{(1)} = \det(\ \langle k, N-k | m, N-m \rangle\ ) \ \Rightarrow \ \text{triangular matrix}$$

where $|m, N-m\rangle = H^m P_0^{N-m} |\delta, p\rangle$: vectors in level N subspace

$\ell = 2$ algebra $\mathfrak{g}_2$

- basic data

$$\mathfrak{g}_2^+ = \langle\ H, P_0, P_1\ \rangle \ \overset{\dagger}{\longleftrightarrow}\ \mathfrak{g}_2^- = \langle\ C, P_3, P_4\ \rangle, \qquad \mathfrak{g}_2^0 = \langle\ D, P_2\ \rangle = (\mathfrak{g}_2^0)^\dagger$$

$$V_2^{\delta,p} = \{\ |k,m,n\rangle = H^k P_0^m P_1^n |\delta, p\rangle\ \}, \quad \text{level } N = k + m + 2n$$

- many null vectors : eg. $P_0^m P_1^n |\delta, p\rangle$

$\Rightarrow$ at least two rows of $\Delta_N^{(2)}$ are proportional $\Rightarrow$ $\Delta_N^{(2)} = 0$

Detection of singular vectors - Kac determinant -

Proposition 1

Kac determinants $\Delta_N^{(\ell)}$ at level N of $\mathfrak{g}_\ell$ (up to overall sign):

$$(i) \quad \Delta_N^{(1)} = \left(\prod_{m=0}^{N} m! \right)^2 (2p)^{N(N+1)}, \quad (\ell = 1)$$

$$(ii) \quad \Delta_N^{(\ell)} = (\ell+1)^2 p^2 \, \delta_{N1}, \quad (\ell \geq 2) \qquad ; \; P_\ell \, |\delta, p\rangle = p \, |\delta, p\rangle$$

distinction between $\ell = 1$ and $\ell \geq 2$

- $\ell = 1 \;\Rightarrow\; \Delta_N^{(1)} \sim p^{N(N+1)}$ for all N
- $\ell \geq 2 \;\Rightarrow\; \Delta_N^{(\ell)} = 0$ for many N

Reducibility of $V_\ell^{\delta,p}$

Proposition 2

1. $V_\ell^{\delta,p}$ is irreducible if $\ell = 1$ and $p \neq 0$.
2. $V_\ell^{\delta,p}$ is reducible if $\ell \geq 2$ and $p \neq 0$.
3. if $p = 0$ then $V_\ell^{\delta,0}$ is reducible for all values of ℓ.

$(\because)$ $^\exists$singular vectors for case 2 and 3

In the following we focus only on $p \neq 0$ case

Formula of singular vectors : part 1

Proposition 3

Followings are singular vectors in $V_\ell^{\delta,p}$ for $\ell \geq 2$.

$$(\mathcal{S}^{(2n)})^k \left|\delta, p\right\rangle, \quad (\mathcal{S}^{(2n+1)})^k \left|\delta, p\right\rangle, \quad n, k = 1, 2, \dots$$

where

$$\mathcal{S}^{(2n)} = pP_{\ell-2n} + \sum_{j=1}^{n-1} a_j P_{\ell-2n+j} P_{\ell-j} + \frac{(-1)^n}{2} \frac{\ell!(\ell+2n)!}{((\ell+n)!)^2} P_{\ell-n}^2,$$

$$\mathcal{S}^{(2n+1)} = p^2 P_{\ell-2n-1} - \frac{\ell+2n+1}{(n+\frac{1}{2})(\ell+1)} P_{\ell-1} S^{(2n)}$$

$$-\frac{n-\frac{1}{2}}{n+\frac{1}{2}} \frac{\ell+2n+1}{\ell+1} pP_{\ell-1}P_{\ell-2n} - \sum_{j=1}^{n-1} b_j P_{\ell-2n+j} P_{\ell-j-1},$$

$(\because)$ check definition by direct computation

Formula of singular vectors : part 1

Examples

• $n = 1$

$$\big(\, 2(\ell+1)pP_{\ell-2} - (\ell+2)P_{\ell-1}^2 \,\big)^k\, |\delta, p\rangle$$

$$\big(\, 3(\ell+1)^2p^2P_{\ell-3} - 3(\ell+3)(\ell+1)pP_{\ell-2}P_{\ell-1} + (\ell+3)(\ell+2)P_{\ell-1}^3 \,\big)^k\, |\delta, p\rangle$$

• $n = 2$

$$\big(\, 2(\ell+2)(\ell+1)pP_{\ell-4} - 2(\ell+4)(\ell+2)P_{\ell-3}P_{\ell-1} + (\ell+4)(\ell+3)P_{\ell-2}^2 \,\big)^k \times |\delta, p\rangle$$

$$\begin{aligned}\big(\, & 5(\ell+2)(\ell+1)^2p^2P_{\ell-5} - 5(\ell+5)(\ell+2)(\ell+1)pP_{\ell-4}P_{\ell-1} \\ & +2(\ell+5)(\ell+4)(\ell+2)P_{\ell-1}^2P_{\ell-3} - (\ell+5)(\ell+4)(\ell+3)P_{\ell-1}P_{\ell-2}^2 \\ & +(\ell+5)(\ell+4)(\ell+1)pP_{\ell-3}P_{\ell-2} \,\big)^k\, |\delta, p\rangle\end{aligned}$$

$$k = 1, 2, 3, \cdots$$

Formula of singular vectors : part 2

Proposition 4

Followings are singular vectors in $V_\ell^{\delta,p}$ for $\ell \geq 2$.

$$(\mathcal{T}^{(2n+1)})^k \, |\delta, p\rangle \,, \quad n, k = 1, 2, \dots$$

where

$$\mathcal{T}^{(2n+1)} = P_{\ell-1}((\ell+2)P_{\ell-1}^2 - 2p(\ell+1)P_{\ell-2})^n + \sum_{j=0}^{n-1} c_j P_{\ell-1}^{2(n-j-1)} P_{\ell-2}^j P_{\ell-3} + \sum_{j=1}^{n} d_j P_{\ell-1}^{2n-2j+1} P_{\ell-2}^j,$$

($\because$) check definition by direct computation

Formula of singular vectors : part 2

Examples

• $n = 1$

$$\big(\, 3(\ell+1)^2 p^2 P_{\ell-3} - 3(\ell+3)(\ell+1) p P_{\ell-2} P_{\ell-1} + (\ell+3)(\ell+2) P_{\ell-1}^3 \,\big)^k \, |\delta, p\rangle$$

• $n = 2$

$$\begin{aligned} \big(\, &(\ell+3)(\ell+2)^2 P_{\ell-1}^5 - 5(\ell+3)(\ell+2)(\ell+1) p P_{\ell-2} P_{\ell-1}^3 \\ &+3(\ell+2)(\ell+1)^2 p^2 P_{\ell-3} P_{\ell-1}^2 + 6(\ell+3)(\ell+1)^2 p^2 P_{\ell-2}^2 P_{\ell-1} \\ &-6(\ell+1)^3 p^3 P_{\ell-3} P_{\ell-2} \,\big)^k \, |\delta, p\rangle \end{aligned}$$

$$k = 1, 2, 3, \cdots$$

Application

formulae of SV $\Rightarrow$ used to derive invariant diff eqs for $\mathfrak{g}_\ell$

How to find invariant differential equations Dobrev 1988

Basic idea

realize $V_\ell^{\delta,p}$ by $\begin{cases} C^\infty\text{-function on } G_\ell \quad (\text{Lie group generated by } \mathfrak{g}_\ell) \\ \text{differential operators} \end{cases}$

Right covariant function $f(g_+)$ on G_ℓ

$$g_\pm = \exp(\mathfrak{g}_\ell^\pm), \quad g_0 = \exp(\mathfrak{g}_\ell^0) = \exp(\alpha D + \beta P_\ell) \quad \in G_\ell$$

$$\Rightarrow \; f(g_+ g_0 g_-) = \exp(\alpha\delta + \beta p) f(g_+)$$

$\downarrow$ action of $X \in \mathfrak{g}_\ell$ defined by $\boxed{Xf(g_+) = \frac{d}{d\tau} f(g_+ e^{\tau X}) \Big|_{\tau=0}}$

- $f(g_+)$: lowest weight vector
- $\mathfrak{g}_\ell^+$: differential operator on $f(g_+)$

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↓ action of $X \in \mathfrak{g}_\ell$ defined by $\boxed{Xf(g_+) = \dfrac{d}{d\tau} f(g_+ e^{\tau X})\Big|_{\tau=0}}$

- $Df(g_+) = \delta f(g_+), \ P_\ell f(g_+) = pf(g_+), \ Xf(g_+) = 0 \ (X \in \mathfrak{g}_\ell^-)$
- $\mathfrak{g}_\ell^+$: differential operator on $f(g_+)$

How to find invariant differential equations Dobrev 1988

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- $Df(g_+) = \delta f(g_+), \; P_\ell f(g_+) = p f(g_+), \; Xf(g_+) = 0 \; (X \in \mathfrak{g}_\ell^-)$
- $\mathfrak{g}_\ell^+$: differential operator on $f(g_+)$ $\rightarrow$ next page

How to find invariant differential equations Dobrev 1988

Parametrization of $g_+ \in G_\ell$

$$g_+ = \exp(tH) \exp\left(\sum_{k=0}^{\ell-1} x_k P_k \right),$$

t, x_k : coordinates on group manifold

$\Rightarrow$ variables of diff eq

Action of $\mathfrak{g}_\ell^+$

$$P_n f(g_+) = \partial_{x_n} f(g_+), \qquad Hf(g_+) = \left(\partial_t + \sum_{k=1}^{\ell-1} k\, x_k \partial_{x_{k-1}} \right) f(g_+)$$

Verma module is realized in space of functions

Verma module	Realization
$\lvert\delta, p\rangle$	$f(g_+)$
$P_1 \lvert\delta, p\rangle$	$\partial_{x_1} f(g_+)$
$P_1 P_2 \lvert\delta, p\rangle$	$\partial_{x_1}\partial_{x_2} f(g_+)$
...	...

Theorem Dobrev 1988

Realization of singular vector has a kernel $\Rightarrow$ invariant equations

Invariant differential equations

Theorem says ...

singular vector $\left(2(\ell+1)pP_{\ell-2} - (\ell+2)P_{\ell-1}^2 \right)^k |\delta, p\rangle$

$$\Big\downarrow \text{realization } \boxed{P_n f(g_+) = \partial_{x_n}}$$

$$\left(2(\ell+1)p\frac{\partial}{\partial x_{\ell-2}} - (\ell+2)\frac{\partial^2}{\partial x_{\ell-1}^2} \right)\psi(x) = 0$$

Invariant differential equations

Theorem says ...

singular vector $\big(2(\ell+1)pP_{\ell-2} - (\ell+2)P_{\ell-1}^2\big)^k |\delta, p\rangle$

$\Big\downarrow$ realization $\boxed{P_n f(g_+) = \partial_{x_n}}$

$$\left(2(\ell+1)p\frac{\partial}{\partial x_{\ell-2}} - (\ell+2)\frac{\partial^2}{\partial x_{\ell-1}^2}\right)\psi(x) = 0$$

Hierarchies of invariant differential equations

Proposition 5

$$\left(p\frac{\partial}{\partial x_{\ell-2n}} + \sum_{j=1}^{n-1} a_j \frac{\partial^2}{\partial x_{\ell-2n+j}\partial x_{\ell-j}} + \frac{(-1)^n}{2}\frac{\ell!(\ell+2n)!}{((\ell+n)!)^2}\frac{\partial^2}{\partial x_{\ell-n}^2}\right)^k \psi(x) = 0$$

Invariant differential equations

Proposition 5 (continue)

$$\left(p^2 \frac{\partial}{\partial x_{\ell-2n-1}} - \frac{\ell+2n+1}{(n+\frac{1}{2})(\ell+1)} \frac{\partial}{\partial x_{\ell-1}} \Big(\sum_{j=1}^{n-1} a_j \frac{\partial^2}{\partial x_{\ell-2n+j} \partial x_{\ell-j}} + \frac{(-1)^n}{2} \frac{\ell!(\ell+2n)!}{((\ell+n)!)^2} \frac{\partial^2}{\partial x_{\ell-n}^2} \Big) \right.$$
$$\left. - \frac{\ell+2n+1}{\ell+1} p \frac{\partial^2}{\partial x_{\ell-1} x_{\ell-2n}} - \sum_{j=1}^{n-1} b_j \frac{\partial^2}{\partial x_{\ell-2n+j} x_{\ell-j-1}} \right)^k \psi(x) = 0,$$

$$\left(\frac{\partial}{\partial x_{\ell-1}} \Big((\ell+2) \frac{\partial^2}{\partial x_{\ell-1}^2} - 2p(\ell+1) \frac{\partial}{\partial x_{\ell-2}} \Big)^n + \sum_{j=0}^{n-1} c_j \Big(\frac{\partial}{\partial x_{\ell-1}} \Big)^{2(n-j-1)} \Big(\frac{\partial}{\partial x_{\ell-2}} \Big)^j \frac{\partial}{\partial x_{\ell-3}} \right.$$
$$\left. + \sum_{j=1}^{n} d_j \Big(\frac{\partial}{\partial x_{\ell-1}} \Big)^{2n-2j+1} \Big(\frac{\partial}{\partial x_{\ell-2}} \Big)^j \right)^k \psi(x) = 0$$

Examples of invariant differential equations

$n = 1$

$$\left(p\frac{\partial}{\partial x_{\ell-2}} - \frac{\ell+2}{2(\ell+1)}\frac{\partial^2}{\partial x_{\ell-1}^2} \right)^k \psi(x) = 0,$$

$$\left(p^2\frac{\partial}{\partial x_{\ell-3}} - \frac{\ell+3}{\ell+1}p\frac{\partial^2}{\partial x_{\ell-2}\partial x_{\ell-1}} + \frac{(\ell+2)(\ell+3)}{3(\ell+1)^2}\frac{\partial^3}{\partial x_{\ell-1}^3} \right)^k \psi(x) = 0,$$

$$\left((\ell+2)\frac{\partial^3}{\partial x_{\ell-1}^3} - 3(\ell+1)p\frac{\partial^2}{\partial x_{\ell-1}\partial x_{\ell-2}} + \frac{3(\ell+1)^2}{\ell+3}p^2\frac{\partial}{\partial x_{\ell-3}} \right)^k \psi(x) = 0.$$

Examples of invariant differential equations

$n = 2$

$$\left(p \frac{\partial}{\partial x_{\ell-4}} - \frac{\ell+4}{\ell+1} \frac{\partial^2}{\partial x_{\ell-3} \partial x_{\ell-1}} + \frac{(\ell+3)(\ell+4)}{2(\ell+1)(\ell+2)} \frac{\partial^2}{\partial x_{\ell-2}^2} \right)^k \psi(x) = 0,$$

$$\left(p^2 \frac{\partial}{\partial x_{\ell-5}} - \frac{\ell+5}{\ell+1} p \frac{\partial^2}{\partial x_{\ell-4} \partial x_{\ell-1}} + \frac{(\ell+4)(\ell+5)}{5(\ell+1)(\ell+2)} p \frac{\partial^2}{\partial x_{\ell-3} \partial x_{\ell-2}} \right.$$
$$\left. + \frac{2(\ell+4)(\ell+5)}{5(\ell+1)^2} \frac{\partial^3}{\partial x_{\ell-3} \partial x_{\ell-1}^2} - \frac{(\ell+3)(\ell+4)(\ell+5)}{5(\ell+1)^2(\ell+2)} \frac{\partial^3}{\partial x_{\ell-2} \partial x_{\ell-1}} \right)^k \psi(x) = 0,$$

$$\left((\ell+2)^2 \frac{\partial^5}{\partial x_{\ell-1}^5} - 5(\ell+1)(\ell+2) p \frac{\partial^4}{\partial x_{\ell-2} \partial x_{\ell-1}^3} + \frac{3(\ell+1)^2(\ell+2)}{\ell+3} p^2 \frac{\partial^3}{\partial x_{\ell-1}^2 \partial x_{\ell 3}} \right.$$
$$\left. + 6(\ell+1)^2 p^2 \frac{\partial^3}{\partial x_{\ell-1} \partial x_{\ell-2}^2} - \frac{6(\ell+1)^3}{\ell+3} p^3 \frac{\partial^2}{\partial x_{\ell-2} \partial x_{\ell-3}} \right)^k \psi(x) = 0.$$

Proof of symmetry

Rep $T^{\delta,p}$ of G_ℓ on the space of right covariant functions

$$(T^{\delta,p}(g)f)(h) = f(g^{-1}h), \quad g, h \in G_\ell$$

Singular vector: $F(h) = \mathcal{P}(\partial_k)f(h), \quad (\mathcal{P}$; differential op)

$\Rightarrow$ another rep of G_ℓ

$$\begin{array}{ccc} (T^{\delta',p'}(g)F)(h) & = & F(g^{-1}h) \\ \| & & \| \\ (T^{\delta',p'}(g)\mathcal{P}f)(h) & & \mathcal{P}f(g^{-1}h) \; = \mathcal{P}(T^{\delta,p}(g)f)(h) \end{array}$$

Intertwining property of $\mathcal{P}$ $\boxed{\mathcal{P}T^{\delta,p} = T^{\delta',p'}\mathcal{P}}$

Differential equation : $\mathcal{P}(\partial_k)\psi(x) = 0$

$$\mathcal{P}(T^{\delta,p}(g)\psi) = T^{\delta',p'}\mathcal{P}\psi = 0, \text{ i.e., symmetry}$$

Irreducible lowest weight modules of $\mathfrak{g}_\ell$

We return to the abstract Verma module $V_\ell^{\delta,p}$,

$$V_\ell^{\delta,p} = \left\{ H^k P_{\ell-1}^{m_1} P_{\ell-2}^{m_2} \cdots P_0^{m_\ell} \left|\delta, p\right\rangle \ \middle|\ k, m_i \in \mathbb{Z}_{\geq 0} \right\}$$

then one can show

Theorem

List of irreducible lowest weight modules of $\mathfrak{g}_\ell$

- $\ell = 1 \ \Rightarrow\ V_1^{\delta,p}$ itself
- $\ell \geq 2 \ \Rightarrow$ quotient module $V_\ell^{\delta,p}/\mathcal{I}_2/\mathcal{I}_3/\cdots/\mathcal{I}_\ell$ with the basis

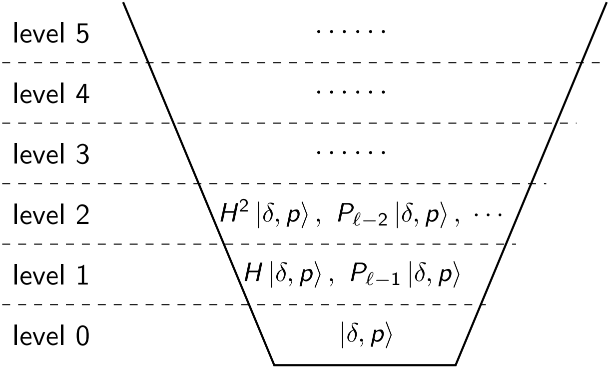
$$H^k P_{\ell-1}^m \left|0\right\rangle \quad \text{where } P_{\ell-a}\left|0\right\rangle \sim P_{\ell-1}^a \left|0\right\rangle, \quad 2 \leq a \leq \ell$$

Cf. R Lü, V Mazorchuk, K Zhao 2014

Irreducible lowest weight modules of $\mathfrak{g}_\ell$

Sketch of proof for $\ell \geq 2$

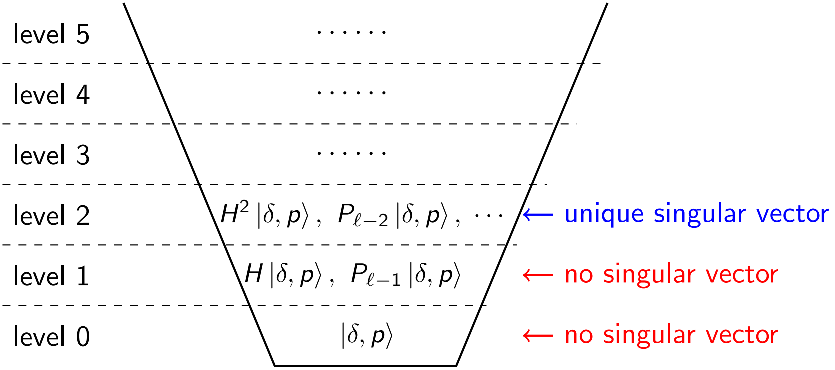
schematic view of $V_\ell^{\delta,p}$



Irreducible lowest weight modules of $\mathfrak{g}_\ell$

Sketch of proof for $\ell \geq 2$

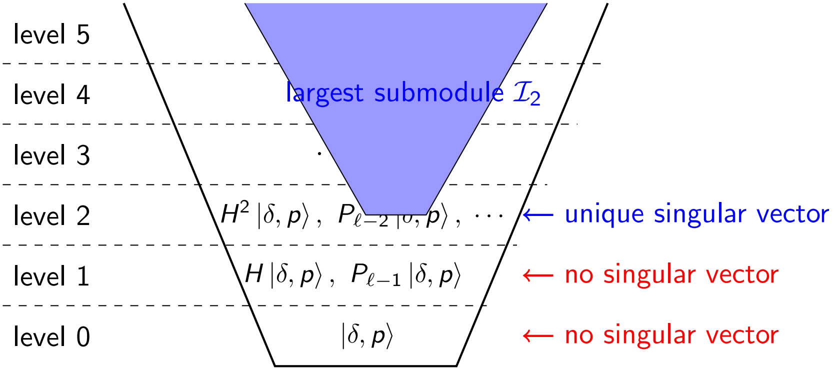
schematic view of $V_\ell^{\delta,p}$



Irreducible lowest weight modules of $\mathfrak{g}_\ell$

Sketch of proof for $\ell \geq 2$

schematic view of $V_\ell^{\delta,p}$

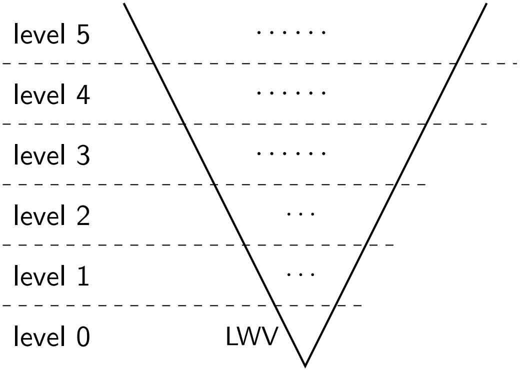


consider $V_\ell^{\delta,p}/\mathcal{I}_2$!

Irreducible lowest weight modules of $\mathfrak{g}_\ell$

Sketch of proof for $\ell \geq 2$

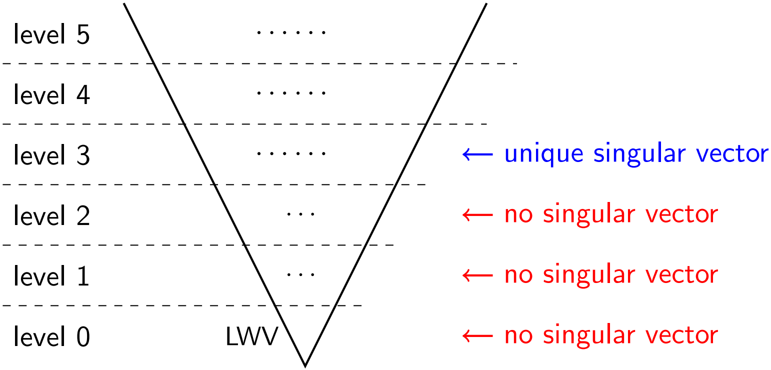
schematic view of $V_\ell^{\delta,p}/\mathcal{I}_2$



Irreducible lowest weight modules of $\mathfrak{g}_\ell$

Sketch of proof for $\ell \geq 2$

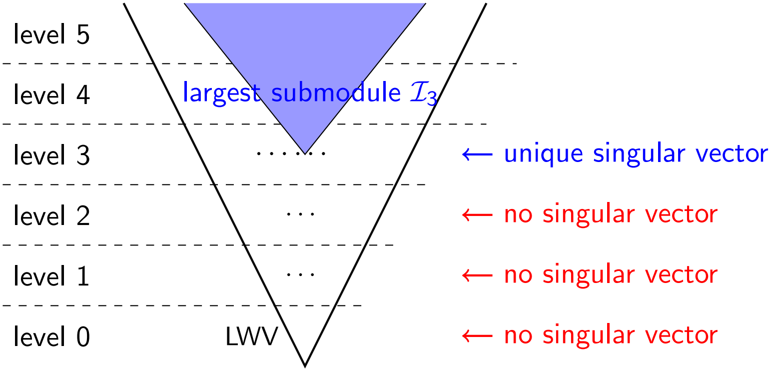
schematic view of $V_\ell^{\delta,p}/\mathcal{I}_2$



Irreducible lowest weight modules of $\mathfrak{g}_\ell$

Sketch of proof for $\ell \geq 2$

schematic view of $V_\ell^{\delta,p}/\mathcal{I}_2$



consider $V_\ell^{\delta,p}/\mathcal{I}_2/\mathcal{I}_3$!

Irreducible lowest weight modules of $\mathfrak{g}_\ell$

Sketch of proof for $\ell \geq 2$

One can repeat this until $V_\ell^{\delta,p}/\mathcal{I}_2/\mathcal{I}_3/\cdots/\mathcal{I}_\ell$

at each step, we have the relation ($\because$ form of singular vector)

$$
\begin{aligned}
V_\ell^{\delta,p}/\mathcal{I}_2 \quad &\Rightarrow \; P_{\ell-2}\,|0\rangle \sim P_{\ell-1}^2\,|0\rangle \\
V_\ell^{\delta,p}/\mathcal{I}_2/\mathcal{I}_3 \quad &\Rightarrow \; P_{\ell-3}\,|0\rangle \sim P_{\ell-1}^3\,|0\rangle \\
V_\ell^{\delta,p}/\mathcal{I}_2/\mathcal{I}_3/\mathcal{I}_4 \quad &\Rightarrow \; P_{\ell-4}\,|0\rangle \sim P_{\ell-1}^4\,|0\rangle \\
\cdots\cdots
\end{aligned}
$$

Summary

Lowest weight reps of $d = 1$, ℓ integer conformal Galilei algebras

- Reducibility of Verma modules $\cdots$ singular vectors
- List of irreducible modules
- differential equations symmetric under the conformal Galilei group

Future works

infinite dimensional version of conformal Galilei algebra

Henkel 1994, Martelli, Bagchi, Gopakumar 2009, Tachikawa 2010, Bagchi *et al* 2010

Hosseiny, Rouhani 2010, 2012, Henkel *et al* 2012, Alishahiha *et al* 2009, Mukhopadhyay 2010

- Virasoro as subalgebra
- relation to vertex operator algebras Gao, Jiang, Pei 2008

Inner product in $V_\ell^{\delta,p}$

algebraic anti-involution $\omega : \mathfrak{g}_\ell \to \mathfrak{g}_\ell$

$$\omega(D) = D, \quad \omega(H) = C, \quad \omega(H) = C, \quad \omega(P_n) = P_{2\ell-n}.$$

$|x\rangle$ and $|y\rangle$: vectors in $V_\ell^{\delta,p}$:

$$|x\rangle = X\,|\delta,p\rangle\,, \qquad |y\rangle = Y\,|\delta,p\rangle\,, \quad X, Y \in U(\mathfrak{g}_\ell^+)$$

Definition

$$\langle x|y\rangle = \langle \delta,p|\,\omega(X)Y\,|\delta,p\rangle\,, \quad \langle \delta,p|\delta,p\rangle = 1.$$

ℓ-Conformal Galilei algebras Negro, del Olmo, Rodriguez-Marco 1997

- $(d+1)$ Dim spacetime $(t, x_1, x_2, \ldots, x_d)$
- 3 sets of generators of transformation

1. Non-relativistic kinematical algebra (Galilei algebra)

$$H = \frac{\partial}{\partial t}, \qquad P_i^{(0)} = \frac{\partial}{\partial x_i}, \qquad P_i^{(1)} = -t\frac{\partial}{\partial x_i}, \qquad M_{ij} \text{ (rotation)}$$

2. Conformal transformation + dilatation

$$C : t \to \frac{t}{1-at}, \quad x_i \to \frac{x_i}{(1-at)^{2\ell}} \quad D : t \to at, \quad x_i \to a^{\ell} x_i$$

3. Additional Abelian generators

$$P_i^{(n)} = (-t)^n \frac{\partial}{\partial x_i}, \qquad n = 2, 3, \cdots, 2\ell, \quad \ell = 1/2, 1, 3/2, \cdots$$

ℓ-Conformal Galilei algebras Negro, del Olmo, Rodriguez-Marco 1997

Non-vanishing commutators

$$[D, H] = 2H, \qquad [D, C] = -2C, \qquad [C, H] = D,$$
$$[M_{ij}, M_{k\ell}] = -\delta_{ik} M_{j\ell} - \delta_{j\ell} M_{ik} + \delta_{i\ell} M_{jk} + \delta_{jk} M_{i\ell},$$
$$[H, P_i^{(n)}] = -nP_i^{(n-1)}, \qquad [D, P_i^{(n)}] = 2(\ell - n)P_i^{(n)},$$
$$[C, P_i^{(n)}] = (2\ell - n)P_i^{(n+1)}, \qquad [M_{ij}, P_k^{(n)}] = -\delta_{ik} P_j^{(n)} + \delta_{jk} P_i^{(n)}.$$

$$i, j = 1, 2, \ldots, d, \quad n = 0, 1, \ldots, 2\ell$$

2 Parameters

- $d = 1, 2, \cdots$ (dim of space), $\ell = 1/2, 1, 3/2, \cdots$
- fix d and ℓ $\Rightarrow$ one ℓ-CGA is defined

Structure

$sl(2, \mathbb{R}) \oplus so(d) \not\ni$ Abelian ideal $P_i^{(n)}$

spin ℓ rep of $sl(2, \mathbb{R})$, vector rep of $so(d)$

Central extensions of ℓ-Conformal Galilei algebras

Central extensions at Abelian sector

$\langle P_i^{(n)} \rangle$ Abelian $\Rightarrow$ non-Abelian

two types of extensions Martelli, Tachikawa 2010

any d and half-integer ℓ (mass extension)

$$[P_i^{(m)}, P_j^{(n)}] = \delta_{ij}\delta_{m+n,2\ell}\, I_m\, M, \quad I_m = (-1)^{m+\ell+1/2}(2\ell - m)!m!$$

only $d = 2$ and integer ℓ (*exotic* extension)

$$[P_i^{(m)}, P_j^{(n)}] = \epsilon_{ij}\delta_{m+n,2\ell}\, I_m\, \Theta,$$
$$I_m = (-1)^m(2\ell - m!)!m!, \quad \epsilon_{12} = -\epsilon_{21} = 1$$

$d = 1$ and half-integer ℓ algebra

Generators

$$H, \; D, \; C, \; M \text{ (center)}, \; P^{(n)} \; (n = 0, 1, \ldots, 2\ell)$$

Non-vanishing commutators

$$[D, H] = 2H, \qquad [D, C] = -2C, \qquad [C, H] = D$$
$$[H, P^{(n)}] = -P^{(n-1)}, \qquad [D, P^{(n)}] = 2(\ell - n)P^{(n)}$$
$$[C, P^{(n)}] = (2\ell - n)P^{(n+1)}, \qquad [P^{(m)}, P^{(n)}] = \delta_{m+n,2\ell} I_m M$$

Triangular decomposition

$$\mathfrak{g}^{+} = \langle \; H, \; P^{(0)}, \ldots, P^{(\ell - \frac{1}{2})} \; \rangle$$
$$\mathfrak{g}^{0} = \langle \; D, \; M \; \rangle$$
$$\mathfrak{g}^{-} = \langle \; C, \; P^{(\ell + \frac{1}{2})}, \ldots, P^{(2\ell)} \; \rangle$$
$$\Rightarrow \; [\mathfrak{g}^{0}, \mathfrak{g}^{\pm}] \subseteq \mathfrak{g}^{\pm}$$

Verma modules over $d = 1$ and half-integer ℓ algebra

Lowest weight vector (vacuum) $|0\rangle$

$$D|0\rangle = -\delta|0\rangle, \quad M|0\rangle = -\mu|0\rangle, \quad X|0\rangle = 0, \ {}^{\forall}X \in \mathfrak{g}^-$$

Verma module $U(\mathfrak{g}^+)|0\rangle$

$$V^{\delta,\mu} = \left\{ H^h \prod_{j=0}^{\ell-\frac{1}{2}} (P^{(\ell-\frac{1}{2}-j)})^{k_j} |0\rangle \ \middle| \ h, k_0, k_1, \cdots, k_{\ell-\frac{1}{2}} \in \mathbb{Z}_{\geq 0} \right\}$$

Singular vectors and irreducible modules

Theorem 1 NA, Isaac, Kimura 2012

$2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 = 0$ $q \in \mathbb{Z}_{\geq 0}$ and $\mu \neq 0$

$\Rightarrow$ $V^{\delta,\mu}$ has precisely one singular vector:

$$|v_q\rangle = \mathcal{S}^q\,|0\rangle\,, \quad \mathcal{S} = 2((\ell - \tfrac{1}{2}))^2\mu H + (P^{(\ell-\frac{1}{2})})^2$$

Theorem 2 NA, Isaac, Kimura 2012

Irreducible lowest weight modules of $d = 1$ ℓ-CGA

(half-integer ℓ, $\mu \neq 0$)

- $V^{\delta,\mu}$ if $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 \neq 0$
- $V^{\delta,\mu}/I^{\delta,\mu}$ if $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 = 0$

All modules are ∞ dim.

Invariant equations

$$\left(2((\ell-\tfrac{1}{2}))^2\mu\Big(\frac{\partial}{\partial t}+\sum_{j=1}^{\ell-\frac{1}{2}} jx_j\frac{\partial}{\partial x_{j-1}}\Big)+\frac{\partial^2}{\partial^2 x_{\ell-\frac{1}{2}}}\right)^q\psi(t,x_i)=0$$

- $\ell = 1/2$ Dobrev, Doebner, Mrugalla 1997

$$\left(2\mu\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}\right)^q\psi(t,x)=0$$

- $\ell = 3/2$

$$\left(2\mu\Big(\frac{\partial}{\partial t}+x_1\frac{\partial}{\partial x_0}\Big)+\frac{\partial^2}{\partial x_1^2}\right)^q\psi(t,x_0,x_1)=0$$

- $\ell = 5/2$

$$\left(8\mu\Big(\frac{\partial}{\partial t}+x_1\frac{\partial}{\partial x_0}+2x_2\frac{\partial}{\partial x_1}\Big)+\frac{\partial^2}{\partial x_2^2}\right)^q\psi(t,x_0,x_1,x_2)=0$$