Poisson vertex algebras and Hamiltonian equations

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slides available at:

`www.mat.uniroma1.it/~desole`

Outline

- Hamiltonian PDEs from an algebraic point of view: Poisson vertex algebras and Poisson structures.
- Lenard-Magri scheme of integrability.
- A 6-parameter family of Poisson structures.
- The corresponding integrable systems.

Joint work with V. Kac and R. Turhan.

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They are evolution equations of the form:

$$\dot{q}_i = \sum_j K_{ij}(q) \cdot \nabla_j h(q) \,,$$

where:

- q_i are local coord on a (fin.dim) manifold M (*phase space*);
- $h(q) \in C^\infty(M)$, is the *Hamiltonian function* (physical "observable");
- $K = (K_{ij}(q))_{i,j=1}^n \in Mat_{n\times n}\mathbb{C}$ is the **Poisson structure** of M (which makes it into a *Poisson manifold*).

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Definition: $K = (K_{ij}(q))_{i,j=1}^n$ is a *Poisson structure* on M means that $C^\infty(M)$ (the algebra of *physical observables*) is a **Poisson algebra** (=Lie algebra + Leibniz rules), with Lie bracket:

$$\{f, g\} = \nabla g \cdot K(q) \nabla f$$

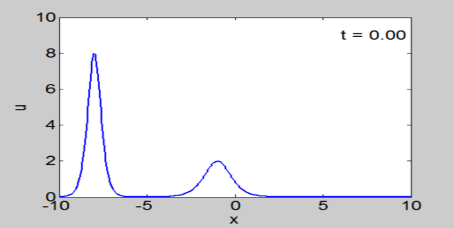
In **Classical Field Theory** (dim $= \infty$): *Hamiltonian PDE*

Esempio: *Korteweg de Vries* (KdV) equation (1895):

$$u_t = 3uu_x + cu_{xxx} ,$$

(it describes the time evolution of waves in shallow waters).
Notation: $u = u(t,x)$, $u_t = \frac{\partial u(t,x)}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, ...
Classical results:

- it is completely integrable;
- it has soliton solutions;
- it is a (bi)**Hamiltonian** equation.



Hamiltonian PDE: they have the form

$$u_t = K(u, u_x, u_{xx}, \dots ; \partial_x) \frac{\delta}{\delta u} \int_M h(u, u_x, u_{xx}, \dots) dx \,,$$

where:

- $u = u(t, x)$, u_x, $u_{xx}, \dots$ are "local coordinates" of an ∞-dimensional space $\mathcal{M} = \mathrm{Fun}(M)$ (*phase space*) of functions on M (*space-time*).
- $\frac{\delta}{\delta u}$ is the *variational derivative* ($u^{(n)} = u_{x \dots x}$):
$$\frac{\delta}{\delta u} = \sum_{n=0}^{\infty} (-\partial_x)^n \frac{\partial}{\partial u^{(n)}} \,.$$
- $\int_M h(u, u_x, u_{xx}, \dots) dx$ is a local functional, or physical "observable" (the *Hamiltonian functional*);
- $K = K(x, u, u_x, \dots ; \partial_x)$ is a differential operator (the **Poisson structure** on $\mathcal{M}$).

Hamiltonian PDE: $u_t = K(\partial_x)\frac{\delta}{\delta u}\int_M h(u, u_x, u_{xx}, \dots)dx$.

Example:

the KdV equation

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Example:

the KdV equation has the Hamiltonian form:

$$\begin{aligned} u_t &= 3uu_x + cu_{xxx} \\ &= \partial_x \frac{\delta}{\delta u}\frac{1}{2}\int dx(u^3 + cuu_{xx}) \end{aligned}$$

∂_x: *Gardner-Faddeev-Zakharov* Poisson structure;

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Example:

the KdV equation has the (bi) Hamiltonian form:

$$\begin{aligned} u_t &= 3uu_x + cu_{xxx} \\ &= \partial_x \frac{\delta}{\delta u}\frac{1}{2}\int dx(u^3 + cuu_{xx}) \\ &= (u_x + 2u\partial_x + c\partial_x^3)\frac{\delta}{\delta u}\frac{1}{2}\int dx\, u^2 \end{aligned}$$

∂_x: *Gardner-Faddeev-Zakharov* Poisson structure;
$u_x + 2u\partial_x + c\partial_x^3$: *Virasoro-Magri* Poisson structure.

Recall (dim$< \infty$): $K = (K_{ij}(q))_{i,j=1}^{n}$ is a Poisson structure on $M^n \Leftrightarrow$
$\Leftrightarrow V = C^\infty(M)$ is a Poisson algebra (=Lie algebra + Leibniz rules),
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$\Leftrightarrow$ $\mathcal{F}=\{\int_M f(u,u_x,\dots)dx\}$, the space of local functionals, is a Lie algebra with Lie bracket

$$\{\int_M f dx, \int g dx\}=\int_M \frac{\delta g}{\delta u}K(\partial_x)\frac{\delta f}{\delta u}$$

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$\Leftrightarrow$ $\mathcal{V}=\{f(u,u_x,\dots)\}$, the algebra of density functions, is a Poisson vertex algebra (=Lie conformal algebra + Leibniz rule).

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  (i) sesquilinearity: $[\partial a_\lambda b] = -\lambda[a_\lambda b]$, $[a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$;
  (ii) skewsymmetry: $[a_\lambda b] = -[b_{-\lambda-\partial} a]$ (∂ is moved to the left);
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- A Poisson vertex algebra is a comm. assoc. differential algebra $\mathcal{V}$ (derivation ∂), with a LCA λ-bracket $\{\cdot\,_\lambda\,\cdot\}$, s.t.:
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- On $\mathcal{V} = \{f(u, u', u'', \dots, u^{(n)})\}$ on $\mathcal{M}$ (endowed with commuting derivations $\frac{\partial}{\partial u^{(n)}}$) a PVA λ-bracket is defined in terms of the corresponding Poisson structure:

$$K(u, u', \dots; \partial) = \{u_\partial u\}_\to$$

  via the so called Master Formula:

$$\{f_\lambda g\}_z = \frac{\partial g}{\partial u^{(n)}} (\lambda + \partial)^n K(\lambda + \partial)(-\lambda - \partial)^m \frac{\partial f}{\partial u^{(m)}}$$

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Example. GFZ: $\{u_\lambda u\} = \lambda$, VirMag: $\{u_\lambda u\} = u' + 2u\lambda$.

Assume: on the algebra of density funct.s $\mathcal{V} = \{f(u, u', \ldots, u^{(n)})\}$ (with partial deriv.s $\frac{\partial}{\partial u^{(n)}}$), we have a PVA λ-bracket $\{\cdot\,_\lambda\,\cdot\}$, (or, equivalen. a Poisson structure $K(\partial)$ on $\mathcal{V}$).

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Consider: the space of local functionals $\mathcal{F} = \mathcal{V}/\partial\mathcal{V}$ (notation: $\int f = \int_M f(u, u_x, \dots) dx$). We have the Lie algebra bracket:

$$\{\textstyle\int f, \int g\} = \int \{f_\lambda g\}|_{\lambda=0} = \displaystyle\int \frac{\delta g}{\delta u} K(\partial) \frac{\delta f}{\delta u}$$

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An integral of motion is a local funct. $\int g \in \mathcal{V}/\partial\mathcal{V}$ s.t.

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GOAL: construct an infinite sequence $\int h_0 = \int h, \int h_1, \int h_2, \dots,$ of lin.indep. integrals of motion in involution:

$$\{\textstyle\int h_m, \int h_n\} = 0 \;\forall m, n \geq 0$$

We then have the integrable hierarchy: $\frac{du}{dt_n} = \{\int h_n, u\}$.

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Assumption: bi-Hamiltonian equation, which can be written in Hamiltonian form in two compatible ways:

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Problem: why can we solve the LM recursion eq. for $\int h_{n+1}$?

Sufficient conditions for the applicability of the Lenard-Magri scheme.

A) *ORTHOGONALITY CONDITION*

Theorem[Barakat, DS, Kac, 2009] Assumptions:

- $\mathcal{V}$: algebra of differential functions (domain and *normal*);
- $K_0, K_1 \in \mathrm{Mat}_{\ell\times\ell}\mathcal{V}[\partial]$: compatible Poisson structures on $\mathcal{V}$,
- K_0: non-degenerate (Dieud.det $K_0 \neq 0$);
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Then: the Lenard-Magri scheme $K_1(\partial)\delta h_n = K_0(\partial)\delta h_{n+1}$ has solution $\int h_{n+1}$ for every n; thus: $\{\int h_n\}_{n=0}^{\infty}$ is an ∞-sequence of integrals of motion in involution: $\{\int h_m, \int h_n\}_{K_0} = 0$, $\forall m, n$ (integrability).

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Corollary[DS, Kac, Turhan, 2013] Assumptions:

- $\mathcal{V}$: algebra of differential functions (domain and *normal*);
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- $C(K_0) \cap C(K_1)$ has codimension 1 in $C(K_0)$.

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- $K_0, K_1 \in \mathrm{Mat}_{\ell\times\ell}\mathcal{V}[\partial]$: compatible Poisson structures on $\mathcal{V}$,
- K_0: strongly skewadjoint;
- $C(K_0) \cap C(K_1)$ has codimension 1 in $C(K_0)$.

Then: letting $\int h_0 \in C(K_0) \cap C(K_1)$ and $\int h_1 \in C(K_0) \backslash C(K_1)$, we have a biHamiltonian equation:

$$\frac{du}{dt} = K_1(\partial)\delta h_0 = K_0(\partial)\delta h_1$$

to which the Lenard-Magri scheme $K_1(\partial)\delta h_n = K_0(\partial)\delta h_{n+1}$ can be successfully applied for every n (integrability).

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Then: for each such pair (K_0, K_1) we automatically get an integrable biHamiltonian PDE:

$$\frac{du}{dt} = K_1(\partial) \frac{\delta h_1}{\delta u} \quad \text{where} \quad \textstyle\int h_1 \in C(K_0) \backslash C(K_1)$$

and the corresponding integrals of motion in involution $\{\int h_n\}_{n \in \mathbb{Z}_+}$ solving $K_0(\partial) \frac{\delta h_{n+1}}{\delta u} = K_1(\partial) \frac{\delta h_n}{\delta u}$.

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- Lie conformal algebra $L = \mathbb{F}[\partial]u \oplus \mathbb{F}[\partial]v$, with λ-brackets

$$\{u_\lambda u\} = (\partial + 2\lambda)u \ , \quad \{u_\lambda v\} = (\partial + \lambda)v \ , \quad \{v_\lambda v\} = 0$$

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(Note: it corresponds to Lie alg. $\mathfrak{g}$ of diff.oper.s of order ≤ 1 on S^1.
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$$\{u_\lambda u\} = (\partial + 2\lambda)u + \alpha\lambda + c\lambda^3 \ ,$$

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- Hence: 6-param.family of (compatible) Poisson structures:

$$K^{(c,a,\gamma)}_{(\alpha,\beta,\varepsilon)}(\partial) = \begin{pmatrix} a(u' + 2u\partial) + \alpha\partial + c\partial^3 & a\,v\partial + \beta\partial + \gamma\partial^2 \\ a\partial \circ v + \beta\partial - \gamma\partial^2 & \varepsilon\partial \end{pmatrix}$$

STEP 2: strongly skewadjoint $K_0(\partial)$.

Two cases to consider separately: **A)** $a \neq 0$; and **B)** $a = 0$.

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Case A): $a \neq 0$
After chg of var.s, can assume $a = 1, \alpha = \beta = 0$. Hence:

$$K(\partial) = K^{(c,1,\gamma)}_{(0,0,\varepsilon)}(\partial) = \begin{pmatrix} u' + 2u\partial + c\partial^3 & v\partial + \gamma\partial^2 \\ \partial \circ v - \gamma\partial^2 & \varepsilon\partial \end{pmatrix} \qquad (1)$$

Notation: $p = \varepsilon c + \gamma^2 \in \mathbb{F}$, and $Q = \varepsilon u - \frac{1}{2}v^2 - \gamma v' \in \mathcal{V}$
Proposition: for K as in (1), we have $\ker K = \frac{\delta}{\delta u} C(K)$, where

$$C(K) = \begin{cases} \mathbb{F}\int v & , \quad \text{if } p \neq 0 \\ \mathrm{Span}_{\mathbb{F}}\{\int v, \int\left(\frac{u}{v} - \frac{c}{2}\frac{(v')^2}{v^3}\right)\} & , \quad \text{if } \varepsilon = \gamma = 0 \\ \mathrm{Span}_{\mathbb{F}}\{\int v, \int(Q^{\frac{1}{2}})\} & , \quad \text{if } p = 0, \varepsilon \neq 0 \end{cases}$$

Corollary: $K(\partial)$ is strongly skew-adjoint iff $p = 0$.

Case B): $a = 0$

5-parameter family with constant coefficients:

$$K(\partial) = K^{(c,0,\gamma)}_{(\alpha,\beta,\varepsilon)}(\partial) = \begin{pmatrix} \alpha\partial + c\partial^3 & \beta\partial + \gamma\partial^2 \\ \beta\partial - \gamma\partial^2 & \varepsilon\partial \end{pmatrix} \qquad (2)$$

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Proposition: K in (2) is non-deg. iff $p \neq 0$ or $q = \alpha\varepsilon - \beta^2 \neq 0$.

In this case it is always strongly skew-adjoint.

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Proposition: for $K(\partial)$ in (2), we have $\ker K = \frac{\delta}{\delta u} C(K)$, where

$$C(K) = \begin{cases} \mathrm{Span}_{\mathbb{F}}\{\int v, \int u\} & , \quad \text{if } p = 0, q \neq 0 \\ \mathrm{Span}_{\mathbb{F}}\{\int f_i\}_{i=1}^4 & , \quad \text{if } p \neq 0 \end{cases}$$

where $f_1 = v$, $f_2 = u$, and f_3, f_4 as follows:

(i) if $q\varepsilon \neq 0$, then

$$\left.\begin{aligned} f_3 &= \Big(\cos\sqrt{\frac{q}{p}}x\Big)u - \Big(\frac{\beta}{\varepsilon}\cos\sqrt{\frac{q}{p}}x + \frac{\gamma}{\varepsilon}\sqrt{\frac{q}{p}}\sin\sqrt{\frac{q}{p}}x\Big)v \\ f_4 &= \Big(\sin\sqrt{\frac{q}{p}}x\Big)u + \Big(\frac{\gamma}{\varepsilon}\sqrt{\frac{q}{p}}\cos\sqrt{\frac{q}{p}}x - \frac{\beta}{\varepsilon}\sin\sqrt{\frac{q}{p}}x\Big)v \end{aligned}\right\}$$

(ii) if $\varepsilon = 0, q \neq 0$, then, $f_3 = 2e^{\frac{\beta}{\gamma}x}u + \Big(\frac{\alpha}{\beta} + \frac{c\beta}{\gamma^2}\Big)e^{\frac{\beta}{\gamma}x}v$, $f_4 = e^{-\frac{\beta}{\gamma}x}v$;

(iii) if $\varepsilon \neq 0, q = 0$, then, $f_3 = -x^2u + \Big(\frac{\beta}{\varepsilon}x^2 - 2\frac{\gamma}{\varepsilon}x\Big)v$, $f_4 = -xu + \frac{\beta}{\varepsilon}xv$;

(iv) if $\varepsilon = q = 0$, then, $f_3 = -xu + \frac{\alpha}{2\gamma}x^2v$, $f_4 = xv$.

STEP 3: Integrable biHamiltonian PDEs.

There are two cases to consider:

A) $K_0(\partial)$ as in (1) (in this case we can take $K_1(\partial)$ as in (2));

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Case A): $K_0(\partial)$ as in (1) and $K_1(\partial)$ as in (2)

Bi-Poisson structure (for $p = \varepsilon c + \gamma^2 = 0$):

$$K_0(\partial) = \begin{pmatrix} u' + 2u\partial + c\partial^3 & v\partial + \gamma\partial^2 \\ \partial \circ v - \gamma\partial^2 & \varepsilon\partial \end{pmatrix}, \quad K_1(\partial) = \begin{pmatrix} \alpha_1\partial + c_1\partial^3 & \beta_1\partial + \gamma_1\partial^2 \\ \beta_1\partial - \gamma_1\partial^2 & \varepsilon_1\partial \end{pmatrix}$$

Recall: $C(K_0) = \mathrm{Span}_{\mathbb{F}}\{\int v, \int h_1\}$ and $C(K_0) \cap C(K_1) = \mathbb{F}\int v$.

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Hence: integrable biHamiltonian PDE: $\frac{du}{dt} = K_1(\partial)\frac{\delta h_1}{\delta u}$, where $\int h_1 \in C(K_0) \setminus C(K_1)$.

Recall: $\int h_1$ is different for $\varepsilon = 0$ or $\varepsilon \neq 0$. Hence: two subcases.

Case A1): $\varepsilon = 0$ ($\int h_1 = \int \left(\frac{u}{v} - \frac{c}{2}\frac{(v')^2}{v^3}\right)$)

$$\begin{aligned}
\frac{du}{dt} &= c\gamma_1 \frac{v^{(iv)}}{v^3} - 9c\gamma_1 \frac{v'''v'}{v^4} - c_1 \frac{v'''}{v^2} + c\beta_1 \frac{v'''}{v^3} - 6c\gamma_1 \frac{v''^2}{v^4} + 42c\gamma_1 \frac{v''v'^2}{v^5} \\
&\quad + 6c_1 \frac{v''v'}{v^3} - 6c\beta_1 \frac{v''v'}{v^4} + 2\gamma_1 \frac{v''u}{v^3} - \gamma_1 \frac{u''}{v^2} - 30c\gamma_1 \frac{v'^4}{v^6} - 6c_1 \frac{v'^3}{v^4} \\
&\quad + 6c\beta_1 \frac{v'^3}{v^5} - 6\gamma_1 \frac{v'^2 u}{v^4} + 4\gamma_1 \frac{u'v'}{v^3} - \alpha_1 \frac{v'}{v^2} + 2\beta_1 \frac{v'u}{v^3} - \beta_1 \frac{u'}{v^2}, \\
\frac{dv}{dt} &= c\varepsilon_1 \frac{v'''}{v^3} - 6c\varepsilon_1 \frac{v''v'}{v^4} + \gamma_1 \frac{v''}{v^2} + 6c\varepsilon_1 \frac{v'^3}{v^5} - 2\gamma_1 \frac{v'^2}{v^3} - \beta_1 \frac{v'}{v^2} + 2\varepsilon_1 \frac{v'u}{v^3} - \varepsilon_1 \frac{u'}{v^2}
\end{aligned}$$

For $\varepsilon_1 = 0$ it is a triangular system (reduces to the scalar case).
For $\varepsilon_1 \neq 0$, with the change of variable: $w = u - \frac{\beta_1}{\varepsilon_1} v - \frac{\gamma_1}{\varepsilon_1} v'$, it is

$$\begin{aligned}
\frac{dw}{dt} &= \frac{1}{\varepsilon_1}\left((c_1\varepsilon_1 + \gamma_1^2)\left(\frac{1}{v}\right)''' + (\alpha_1\varepsilon_1 - \beta_1^2)\left(\frac{1}{v}\right)'\right) \\
\frac{dv}{dt} &= -c\varepsilon_1 \frac{1}{v}\left(\frac{1}{v}\right)''' - \varepsilon_1 \left(\frac{w}{v^2}\right)'
\end{aligned}$$

Remarks:

- For $c = 0$ it is the Antonowicz-Fordy equation (1988).
- For $c \neq 0$, seems to be new integrable biHamiltonian PDE.

Case A2: $\varepsilon \neq 0$ ($\int h_1 = \int Q^{\frac{1}{2}}$).

In the new variable $w = Q^{-\frac{1}{2}}, v$, we get the equation

$$\begin{aligned} \frac{dw}{dt} &= (\gamma^2\varepsilon_1 - 2\varepsilon\gamma\gamma_1 - \varepsilon^2 c_1) w^3 w''' \\ &+(\varepsilon\gamma_1 - \gamma\varepsilon_1)(w^4 v'' + 2w^3 w' v') - \varepsilon^2 \alpha_1 w^3 w' \\ &-\varepsilon_1 (w^4 v v' + w^3 w' v^2) + \varepsilon\beta_1 (w^4 v' + 2w^3 w' v) \\ \frac{dv}{dt} &= 2(\gamma\varepsilon_1 - \varepsilon\gamma_1) w'' - 2\varepsilon_1 (w' v + w v') + 2\varepsilon\beta_1 w' . \end{aligned}$$

Remarks

- for $\gamma^2\varepsilon_1 - 2\varepsilon\gamma\gamma_1 - \varepsilon^2 c_1 = 0$ it appeared in the list of [Mikhailov, Shabat, Yamilov, 87]
- in general, seems a new integrable biHamiltonian PDE.

Case B): $K_0(\partial)$ as in (2) and $K_0(\partial)$ as in (1)

Bi-Poisson structure:

$$K_0(\partial) = \begin{pmatrix} \alpha\partial + c\partial^3 & \beta\partial + \gamma\partial^2 \\ \beta\partial - \gamma\partial^2 & \varepsilon\partial \end{pmatrix}, \quad K_1(\partial) = \begin{pmatrix} u' + 2u\partial + c_1\partial^3 & v\partial + \gamma_1\partial^2 \\ \partial \circ v - \gamma_1\partial^2 & \varepsilon_1\partial \end{pmatrix}$$

Notation: $p = \varepsilon c + \gamma^2$, $q = \alpha\varepsilon - \beta^2$. Assumption: $(p, q) \neq (0, 0)$.

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Observation: $C(K_0) \cap C(K_1) = \mathbb{F}\int v$; hence, the codimension 1 property holds iff $\dim C(K_0) = 2$, i.e. for $p = 0$ and $q \neq 0$.

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In this case: we get the following integrable biHamiltonian equation: $\frac{\delta u}{\delta t} = K_1(\partial)\frac{\delta h_1}{\delta u}$, where $\int h_1 = \int u$.

$$\begin{aligned} \frac{du}{dt} &= (c\gamma_1 - c_1\gamma)v^{(iv)} + cv'''v - \beta c_1 v''' + (\varepsilon c_1 + \gamma\gamma_1)u''' \\ &\quad - 2\gamma v''u + \alpha\gamma_1 v'' + \gamma u''v - \beta\gamma_1 u'' - \gamma v'u' + \alpha v'v \\ &\quad - 2\beta v'u - 2\beta vu' + 3\varepsilon u'u\,, \\ \frac{dv}{dt} &= (c\varepsilon_1 + \gamma\gamma_1)v''' - \gamma v''v + \beta\gamma_1 v'' + (\gamma\varepsilon_1 - \varepsilon\gamma_1)u'' - \gamma v'^2 \\ &\quad - 2\beta v'v + \varepsilon v'u + \varepsilon vu' + \alpha\varepsilon_1 v' - \beta\varepsilon_1 u'\,. \end{aligned}$$

Remarks

- For $c = \gamma = 0$, $\varepsilon c_1 \neq 0$ it reduces, after a change of variables, to Kupershmidt equaiton (1985);
- for $c = 0$, $\varepsilon \neq 0$, $c_1 = 0$, and for $c = 0$, $\varepsilon = 0$, it reduces, after potentiation: $u = w'$, to equations in the list of [Mikhailov, Shabat, Yamilov, 87];
- for $c = 0$, $\varepsilon = 0$, $c_1 = 0$ it reduces, after a change of variable, to the Kaup-Broer equation (1985);
- in general: seems a new integrable biHamiltonian PDE.

Remark on the usefulness of PVAs and λ-brackets

As far as "local" Hamiltonian equation are involved, PVAs and λ-brackets are a convenient tool (to simplify notation and computations), but not indispensable.

When considering non-local Hamiltonian equations (with pseudo-differential Poisson structures), they become indispensable: the "rigorous" definition is in terms of λ-brackets.

Further developments (and applications of PVAs)

- variational complex [DS, Kac 08], [DS, Hekmati, Kac, 10]
- variational Poisson cohomology [DS, Kac, 12 and 13]
- W-algebras and Drinfeld-Sokolov biHamiltonian hierarchies [DS, Kac, Valeri, 12 and 13]
- Non-local Poisson structures and non-local Hamiltonian equations [DS, Kac, 13]
- Dirac reduction [DS, Kac Valeri, 13]
- Non-commutative Ham. eq.s [DS, Kac, Valeri, in progress]

The End