Noncommutative (=NC) Instantons and Reciprocity

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Based on

- **MH&T.Nakatsu, ADHM Construction and Group Actions for NC Instantons, to appear (Moyal-product formalism)**
- **MH&T.Nakatsu, Gauge instantons in NC space, work in progress, … (operator formalism)**

cf.**MH&Nakatsu, ADHM const. of NC Instantons, arXiv:1311.5227,…**

1. Introduction

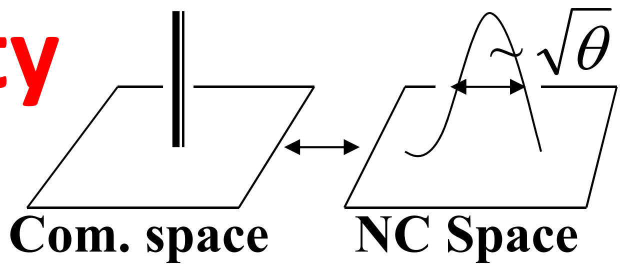
- Non-Commutative (NC) spaces are defined by noncommutativity of spatial coordinates:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$ $\theta^{\mu\nu}$: NC parameter (real const.)

(cf. CCR in QM : $[q, p] = i\hbar$)

(→ ``space-space uncertainty relation'' →)

Resolution of singularity



(→ new physical objects)

Ex) Resolution of small instanton singularity

(→ U(1) instantons) [Nekrasov-Schwarz]

1. Introduction

Anti-Self-Dual Yang-Mills (ASDYM) eqs. play important roles in elementary particle theory, geometry and integrable systems.

- **Finite-action solutions (instantons) reveal non-perturbative effects in QFT ←ADHM is essential.**
- **Noncommutative instantons are used to obtain the Nekrasov partition fcn.**
- **We want to understand the ADHM fully.**
- **(NC extension ←→ background flux)**

ASDYM eq. in 4-dim. with G=U(N)

- **ASDYM eq. (real rep.)** $\mu,\nu=1,2,3,4$

$$F_{12}=-F_{34},\quad F_{13}=-F_{42},\quad F_{14}=-F_{23}.$$

$$F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+A_\mu A_\nu-A_\nu A_\mu$$

Field strength

A_μ: **Gauge field (N × N anti-Hermitian)**

- **There are two descripitions of NC extension:**
  - **Moyal-product formalism** (deformation quantization)
  - **Operator formalism** (Connes' theory)

NC ASDYM eq. with G=U(N) in Moyal

- **NC ASDYM eq. (real rep.)**

$$F_{01}^{*} = -F_{23}^{*},$$
$$F_{02}^{*} = -F_{31}^{*},$$
$$F_{03}^{*} = -F_{12}^{*}$$

$$(F_{\mu\nu}^{*} := \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu * A_\nu - A_\nu * A_\mu)$$

$$\theta^{\mu\nu} = \left[\begin{array}{cc|cc} 0 & -\theta^1 & & \\ \theta^1 & 0 & & O \\ \hline & & 0 & -\theta^2 \\ & O & \theta^2 & 0 \end{array}\right]$$

(Spell: All products are Moyal products.)

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x)$$
$$= f(x)g(x) + i\frac{\theta^{\mu\nu}}{2}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2)$$

Under the spell, we get a theory on NC spaces:

↓

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

Note: Coordinates and functions themselves are c-number-valued usual ones

Under the spell, the solution is deformed:

$$A(x,\theta) = A^{(0)}(x) + \theta A^{(1)}(x) + \theta^2 A^{(2)}(x) + \cdots,$$

2. Atiyah-Drinfeld-Hitchin-Manin Construction

based on reciprocity for the instanton moduli space

4dim. ASDYang-Mills eq.
(Difficult)

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1 z_2}=0$$

N × N PDE

ADHM eq. (≒0dim. ASDYM)
(Easy)

$$[B_1,B_1^{+}]+[B_2,B_2^{+}]+I\,I^{+}-J^{+}J=0$$

$$[B_1,B_2]+IJ=0$$

k × k Matrix eqs.

1:1

Sol.= instantons
(G=U(N), C_2 =k)

↔

Sol.=ADHM data
(G=\`U(k)')

$$A_\mu : N\times N$$

$$B_{1,2} : k\times k,\quad I : k\times N,\quad J : N\times k$$

Gauge trf.:

$$A_\mu \mapsto g^{-1}A_\mu g+g^{-1}\partial_\mu g$$

$$g\in U(N)$$

Gauge trf.:

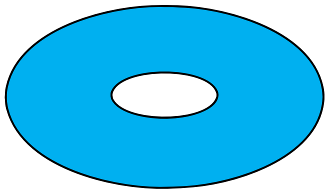
$$B_{1,2}\mapsto \tilde{g}^{-1}B_{1,2}\tilde{g},\quad \tilde{g}\in U(k)$$

$$I\mapsto \tilde{g}^{-1}I,\quad J\mapsto J\tilde{g}$$

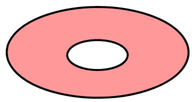
Fourier-Mukai-Nahm transformation

Beautiful reciprocity between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASD Yang-Mills eq. on a 4-torus		4dim. ASD Yang-Mills eq. on the dual torus
$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$ $F_{z_1 z_2} = 0$ N × N PDE	F: ← G: →	$\widetilde{F}_{\xi_1\bar{\xi}_1} + \widetilde{F}_{\xi_2\bar{\xi}_2} = 0$ $\widetilde{F}_{\xi_1\xi_2} = 0$ k × k PDE
Sol.=instantons (G=U(N), C_2 =k)	1:1	Sol.=the dual instantons (G=U(k), C_2=N)
$A_\mu : N \times N$	↕	$\widetilde{A}_\mu : k \times k$



Define the maps F & G,
& G∘F=id. & F∘G=id.



On a 4-torus | **On the dual 4-torus**

Fourier-Mukai-Nahm transformation

Beautiful reciprocity between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq. on a 4-torus

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

N × N PDE

1:1

4dim. ASD Yang-Mills eq. on the dual torus

$$\widetilde{F}_{\xi_1\bar{\xi}_1} + \widetilde{F}_{\xi_2\bar{\xi}_2} = 0$$
$$\widetilde{F}_{\xi_1\xi_2} = 0$$

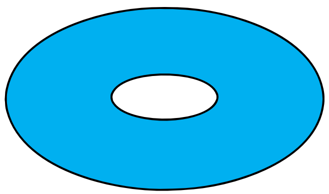
k × k PDE

Sol.=instantons (G=U(N), C_2 =k) ⟷ Sol.=the dual instantons (G=U(k), C_2=N)

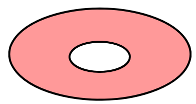
$$A_\mu(x) = \left\langle V, \partial_\mu V \right\rangle_\xi$$

map F (Dirac eq.) ⟵

$$\widetilde{A}_\mu(\xi) : k \times k$$



$$\nabla^+ V = e^\mu \otimes \left(\frac{\partial}{\partial \xi^\mu} + \widetilde{A}_\mu - i x_\mu\right) V = 0$$



$$V : 2k \times \underline{N}$$

On a 4-torus $: x_\mu$

Family index thm.

On the dual 4-torus $: \xi_\mu$

Fourier-Mukai-Nahm trf. (radii of the torus→∞)

Reciprocity between instanton moduli on $\mathbb{R}^4$ and instanton moduli on ``1pt.''

[cf. van Baal, hep-th/9512223]

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

1:1

0dim. ASD Yang-Mills eq.

$$\tilde{F}_{\mu\nu} := \cancel{\partial_\mu} \tilde{A}_\nu - \cancel{\partial_\nu} \tilde{A}_\mu + [\tilde{A}_\mu, \tilde{A}_\nu]$$

$$\tilde{F}_{\xi_1\bar{\xi}_1} + \tilde{F}_{\xi_2\bar{\xi}_2} = 0$$

$$\tilde{F}_{\xi_1\xi_2} = 0$$

Matrix eq. !

k × k ~~PDE~~

Sol.=instantons
(G=U(N), C_2 =k)

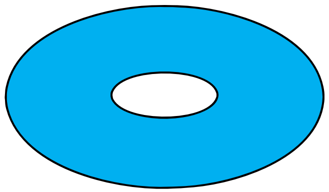
↔

Sol.=``dual instantons''
(G=U(k), ``C_2=N'')

$$A_\mu = V^+ \partial_\mu V$$

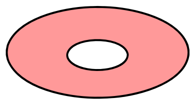
map F (0dim Dirac eq.) ←

$$\tilde{A}_\mu : k \times k$$



$$\nabla^+ V = e^\mu \otimes \left(\cancel{\frac{\partial}{\partial \xi^\mu}} + \tilde{A}_\mu - i x_\mu\right) V = 0$$

Matrix eq. !



$$V : 2k \times \underline{N}$$

Linear alg.

On a 4-torus→R^4

On the dual 4-torus→1 pt.

Atiyah-Drinfeld-Hitchin-Manin (ADHM) Construction

based on the following reciprocity

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

ADHM dim. (≒0dim. ASDYM)

RHS is in fact $[z_1, \bar{z}_1] + [z_2, \bar{z}_2]$

$$[B_1, B_1^+] + [B_2, B_2^+] + I\,I^+ - J^+ J = 0$$

$$[B_1, B_2] + IJ = 0$$

k × k matrix eq.

F(0dim D.eq.) ←

→ G(4dim D.eq.)

1:1 ↔

Proved in the same way as the Nahm trf.

Sol.=instantons (G=U(N), C_2 =k)

$$A_\mu : N \times N$$

Sol.=ADHM data (G=`U(k)')

$$B_{1,2} : k \times k,$$

$$I : k \times N, \quad J : N \times k$$

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) Commutative BPST instanton（N=2, k=1）

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

ADHM eq.(≒0dim. ASDYM)

$$\mu_R = [B_1, B_1^+] + [B_2, B_2^+] + I\,I^+ - J^+ J = 0$$

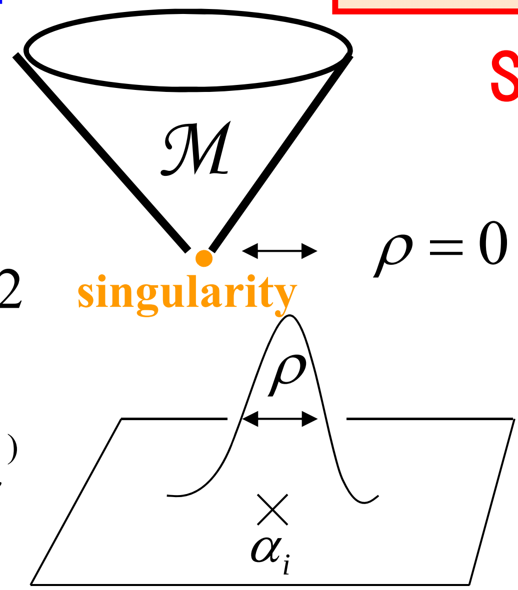
$$\mu_C = [B_1, B_2] + IJ = 0$$

k × k matrix eq.

BPST instanton
(G=U(2), C2 =1)

$$A_\mu = \frac{i(x-b)^\nu \eta^{(-)}_{\mu\nu}}{(z-\alpha)^2 + \rho^2}, \quad 2\times 2$$

$$F_{\mu\nu} = \frac{2i\rho^2}{((z-\alpha)^2+\rho^2)^2}\eta^{(-)}_{\mu\nu}$$



$\rho = 0$

$\rho \to 0$: singular

Sol.=ADHM data
(G=`U(1)’)

position

$$B_{1,2} = \alpha_{1,2}, \quad 1\times 1$$

$$I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

size

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) NC BPST instanton（N=2, k=1）

NC ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

NC ADHM eq.

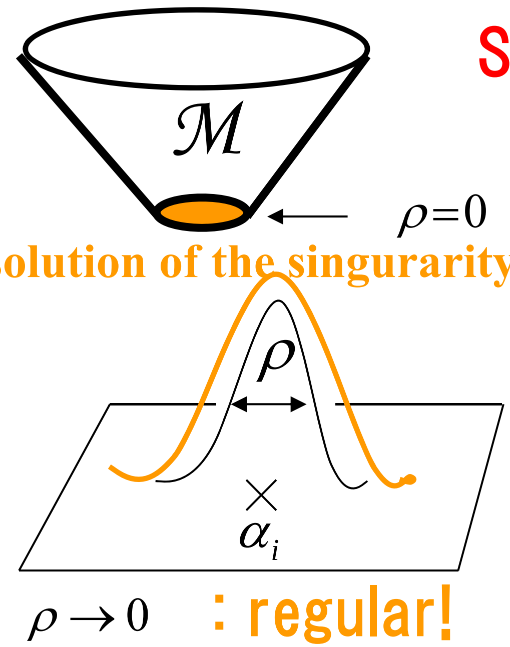
$$\mu_R = [B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = \zeta$$

$$\mu_C = [B_1, B_2] + IJ = 0$$

k × k marix eq.

NC BPST instanton
(G=U(2), C_2 =1)

$A_\mu, F_{\mu\nu}$: exact sol.



Sol.: ADHM data
(G=`U(1)')

position

$$B_{1,2} = \alpha_{1,2},$$

$$I = (\sqrt{\rho^2 + \zeta}, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

size

Fat by ζ !

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) NC BPST instanton（N=2, k=1）

NC ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

N × N PDE

NC ADHM eq.

$$\mu_R = [B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = \zeta$$
$$\mu_C = [B_1, B_2] + IJ = 0$$

k × k matrix eq.

NC BPST instanton
(G=U(2), C_2=1)

↑ By calculation of TrFΛF

$A_\mu, F_{\mu\nu}$: exact sol.

Sol.=ADHM data
(G=`U(1)’)

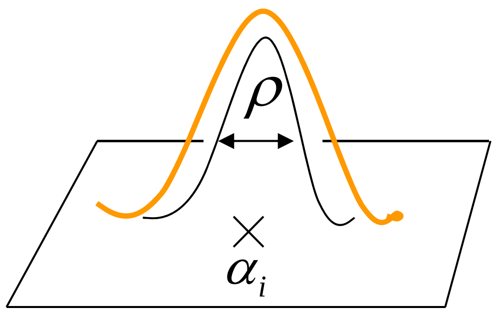
position

$$B_{1,2} = \alpha_{1,2},$$

$$I = (\sqrt{\rho^2 + \zeta}, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

size Fat by ζ !

Do k×k ADHM data give
Instanton number k
in general ? (We prove this.)



$\rho \to 0$: regular!

3. Origin of instanton number in ADHM construction

We prove the formula:

Ex) $f = ``\frac{1}{r^2+\rho^2+\zeta}"(U(2), k=1)$

$$\int d^4x Tr_N F^*_{\mu\nu} * F^*_{\mu\nu} = -\int d^4x \partial^2 \partial_\mu Tr_k f^{-1} * \partial^\mu f$$

$$(\xrightarrow{\theta \to 0} Tr_N F_{\mu\nu} F^{\mu\nu} = -\partial^2 \partial^2 \log \det f)$$

[Corrigan–Goddard–Osborn–Templeton]

$$f := (\nabla^+ * \nabla)^{-1} \qquad \nabla = \begin{pmatrix} I^+ & J \\ \bar{z}_2 - B_2^+ & -(z_1 - B_1) \\ \bar{z}_1 - B_1^+ & z_2 - B_2 \end{pmatrix}$$

: determined by the ADHM data only

: f always exists.

[Nakajima, Maeda–Sako]

Then:

$$C_2 = \frac{1}{16\pi^2} \int d^4x Tr_N F^*_{\mu\nu} * F^*_{\mu\nu} = \frac{1}{16\pi^2} \int d^4x \partial^2 \partial_\mu Tr_k f^{-1} * \partial^\mu f$$

$$= \frac{8}{16\pi^2} \int d\Omega Tr_k \underline{1_k} = k$$

$$f_{k\times k} \approx r^{-2} \cdot 1_k (r \to \infty)$$

comes from the size of the ADHM data!

4. Proof of the reciprocity: (inst)↔(ADHM)

NC instanton $\underset{G}{\overset{F}{\rightleftarrows}}$ NC ADHM

$A_\mu : N \times N$

$B_{1,2} : k \times k,$

$I : k \times N, \quad J : N \times k$

(i) ASD (ASDYM)
(ii) C_2=k
~~**(iii) 𝔇^2 has inverse**~~

(i) ASD (ADHM eq.)
(ii) matrix size = k, N
~~**(iii) ∇^2 has inverse**~~

(iii) is automatically satisfied in the noncommutative situation
[Maeda-Sako] [Nakajima]

Proof of the one-to-one ⇔ Define the maps F & G,
& G∘F=id. & F∘G=id.

F: (ADHM)→(inst):ADHM construction

NC instanton ← 0dim. Dirac eq. — NC ADHM

$$\nabla^{+} * V = 0, \quad V^{+} * V = 1_N$$

$$A_\mu = V^{+} * \partial_\mu V : N \times N$$

$$\nabla = \begin{pmatrix} I^{+} & J \\ \bar{z}_2 - B_2^{+} & -(z_1 - B_1) \\ \bar{z}_1 - B_1^{+} & z_2 - B_2 \end{pmatrix}$$

0 dim *Dirac op*. $(N + 2k) \times 2k$

$$B_{1,2} : k \times k,$$

$$I : k \times N, \quad J : N \times k$$

(i) ASD (ASDYM) [Nekasov-Schwarz]

(ii) C_2=k ←**[MH Nakatsu]**

(i) ASD (ADHM eq.)

(ii) matrix size= k, N

G : (inst)→(ADHM): inverse construction

NC instanton —(4dim. Dirac eq.)→ NC ADHM

$$\bar{e}_\mu D_\mu * \psi = 0, \quad \int d^4 x \psi^+ * \psi = 1_k$$

$A_\mu : N \times N$ $\qquad e^\mu D_\mu : 4\,\mathrm{dim}\ Dirac\ op.$

$$B_{1,2} = \int d^4 x\, z_{1,2} * \psi^+ * \psi : k \times k,$$

$$\psi \approx \frac{I^+, J}{r^3} : N \times k$$

(i) ASD (ASDYM)
(ii) C_2=k

(i) ASD (ADHM eq.)
(ii) matrix size= k, N

[Maeda-Sako2009] proves the existense of the Dirac zero-mode be a formal power expansion of θ recursively.

$$\psi(x,\theta) = \psi^{(0)} + \theta \psi^{(1)} + \theta^2 \psi^{(2)} + \cdots,$$

$$A(x,\theta) = A^{(0)} + \theta A^{(1)} + \theta^2 A^{(2)} + \cdots,$$

G∘F=id : (ADHM)→(inst)→(ADHM)

NC ADHM —(0dim. Dirac eq. V)→ NC instanton —(4dim. Dirac eq. ψ)→ NC ADHM

$B_{1,2}, I, J$

$A_\mu = V^+ * \partial_\mu V$

$B'_{1,2} = B_{1,2}$?

$I' = I$?, $J' = J$?

find
in terms of
the original B, I, J, V
to give the new data

$$\psi = \psi(B, I, J, V)$$

$$= V^+ * Cf$$ **The answer**

$$\bar{e}_\mu D_\mu * \psi = 0, \quad \int d^4x \psi^+ * \psi = 1_k$$

$$B'_{1,2} = \int d^4x\, z_{1,2} * \psi^+ * \psi = \cdots = B_{1,2}$$

$$\psi \approx \frac{I'^+, J'}{r^3} = \cdots = \frac{I^+, J}{r^3}$$

are shown [Maeda-Sako]

F∘G=id : (inst)→(ADHM)→(inst)

NC instanton → (4dim. Dirac eq. ψ) → NC ADHM → (0dim. Dirac eq. V) → NC instanton

A_μ $\quad$ $B_{1,2}, I, J$ $\quad$ $A'_\mu = A_\mu$?

find V in terms of the original instanton Aμ to give the new instanton

$$V = V(A_\mu, \psi)$$

$$D^2 * V = -4\psi^+ C$$ **:the answer**

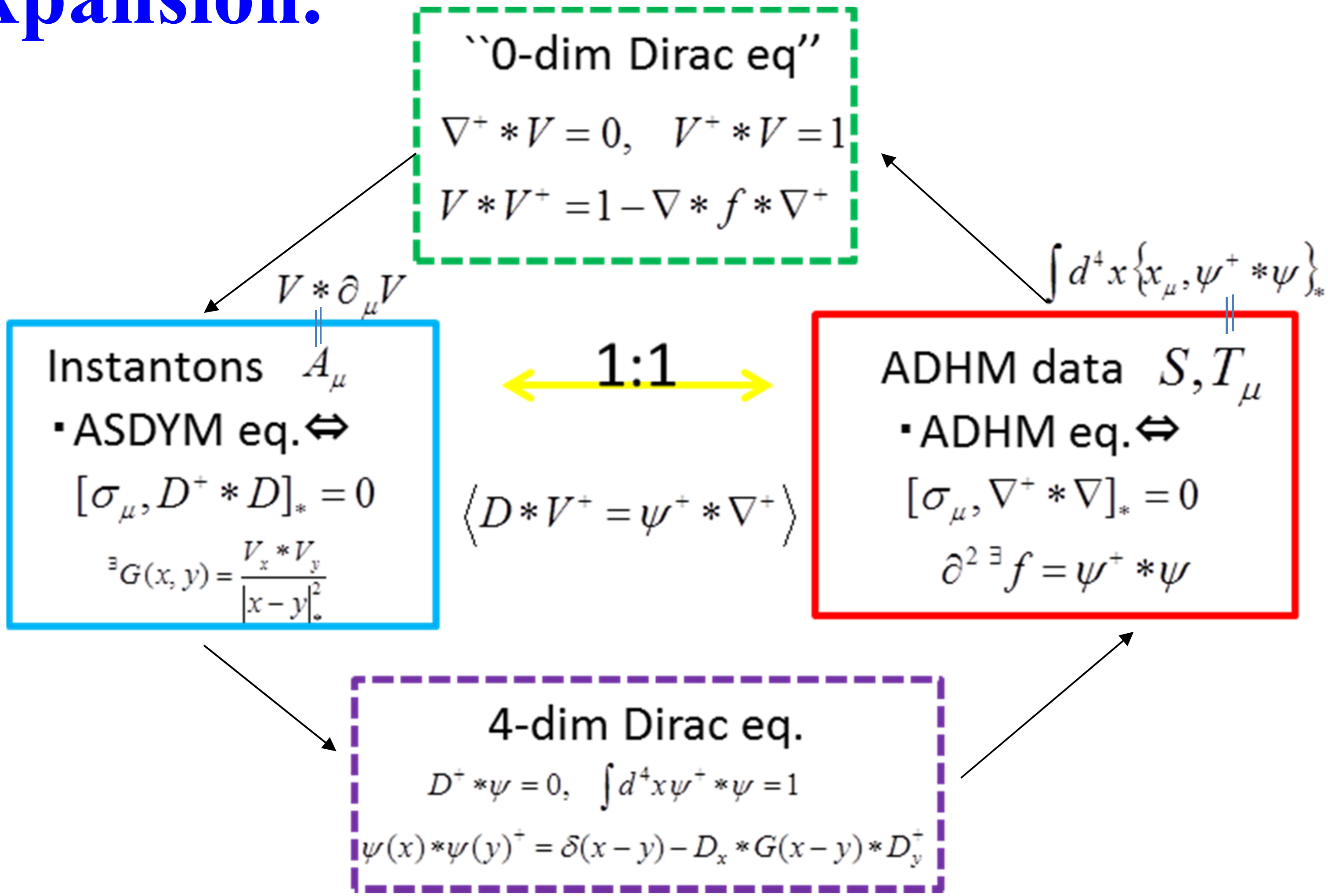
[Maeda-Sako] assume the existence of V.
[MH-Nakatsu] prove it

$$\nabla^+ * V = 0, \quad V^+ * V = 1_N$$

$$A'_\mu = V^+ * \partial_\mu V = \cdots = A_\mu$$

are shown (existence proof is also made by us)

Main result: We prove the ADHM one-to-one correspondence in the formal power series of θ-expansion.



5. Conclusion and Discussion

- We prove the one-to-one correspondence between NC instantons and NC ADHM data in the Moyal-product formalism. [to appear soon…]
- This is valid only in the region that the θ-expansions converge.
- We proceed to reveal the reciprocity in operator formalism. (mostly completed [work in progress])
- We are also interested in non-finite action solutions (soliton-like) of NC ASDYM (twistor theory and Ward's conejecture) in relation to lower-dimensional integrable systems such as KdV. [See e.g. MH, arXiv:1101.0005] (and perhaps the AGT corresp.)