Constructing the Quantum Hall System on the Grassmannians $\mathbf{Gr}_2(\mathbb{C})^N$

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Timeline and Motivations

QHE on S^2

QHE on S^4

QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ *and Generalization to* $\mathbf{Gr}_2(\mathbb{C}^N)$

Local Form of the Wave Functions on $\mathbf{Gr}_2(\mathbb{C}^4)$

Remarks and Conclusions

A Short Timeline

- Shortly after the experimental discovery of QHE by von Klitzing (1980), Laughlin (1981) gave a theoretical description of the effect via a multi-particle wave function describing the $2D$ electron system as a quantum incompressible liquid. Laughlin's description breaks translational invariance.
- Haldane (1983) considered QHE on S^2. Why and how?
- Hu and Zhang (2000) obtained a generalization of QHE on S^4. How? What are the motivations?
- Karabali and Nair (2002) formulated QHE on $\mathbb{C}P^N$.
- QHE systems on S^3, $\mathbb{R}^4$, a Flag Manifold, and others based on higher-dimensional fuzzy spheres were considered in the recent past.
- We formulate QHE on $\mathbf{Gr}_2(\mathbb{C}^{\mathbb{N}})$

Review of QHE on S^2

- Why would we ever want to do this?
  - Rotational Invariance, leading to invariance under magnetic translations, explaining the degeneracy of LL.
  - No edges, enables us to concentrate on the bulk properties, important in understanding some properties of FQHE.
  - Compact geometry leads to finite number of degrees of freedom, leading to finite degeneracy of states at a given LL.
  - When $r \to \infty$, results on the plane are recovered.
- We need a B-field, with fixed magnitude on the surfaces of S^2. We can take a Dirac monopole placed at the center of S^2.

$$\vec{B} = \frac{g}{r^3}\vec{r}, \quad eg = \frac{1}{2}n\hbar$$

- Recall the Hopf Fibration $U(1) \longrightarrow SU(2) \longrightarrow S^2$
  - On $SU(2)$ we have Wigner functions $\mathcal{D}^{(j)}_{L_3,R_3}(g)$. Functions on S^2 may be viewed as the subset of functions on $SU(2)$ invariant under the $U(1)$ subgroup.

- On $SU(2)$ we have left-invariant and right-invariant rotations, they fulfill the $SU(2)$ commutation relations

$$[L_i\,,L_j] = -\varepsilon_{ijk}L_k\,,\quad [R_i\,,R_j] = \varepsilon_{ijk}R_k\,,\quad [L_i\,,R_j] = 0\,.$$
$$R_\pm = R_1 \pm iR_2\,,\quad [R_+\,,R_-] = 2R_3\,,\quad [R_\pm\,,R_3] = \mp R_\pm\,,$$

- Trivial right action of $U(1)$: $R_3 = 0$ gives $\mathcal{D}^l_{m,0} = \sqrt{\frac{4\pi}{(2l+1)}}Y^*_{lm}(\theta,\phi)$
- Covariant derivatives on S^2 are $D_\pm = i\frac{1}{\sqrt{2}r}R_\pm$. Observe that $[D_+,D_-] = B$, where $B = \frac{n}{2r^2} = \frac{2}{r^2}R_3$
- Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2m}\left(D_+D_- + D_-D_+\right) \\ &= \frac{1}{2mr^2}\left(\sum_{i=1}^{3} R_i^2 - R_3^2\right) \end{aligned}$$

- Eigenvalue equation for H is solved by the wave functions (in units where $\hbar = 1$ and $e = 1$.)

$$\mathcal{D}^j_{L_3\,\frac{n}{2}}\,,\quad R_3 = \frac{n}{2}\,,\quad j = q + \frac{n}{2}\,,$$

- Spectrum of the Hamiltonian reads

$$\frac{1}{2mr^2}\left(\frac{n}{2}(2q+1)+q(q+1)\right)=\frac{B}{2m}(2q+1)+\frac{1}{2mr^2}q(q+1)\,,$$

- Observe that $[L_i\,,H]=0$. Therefore, degeneracy of the eigenstates is given by the possible values of L_3.
- In a given LL there is $2j+1=n+1+2q$ fold degeneracy controlled by the eigenvalues of L_3.
- We can write down the many particle wave functions say in the LLL.
- It is possible to argue that the QHE system forms an incompressible liquid. Under fixed B-field, each Landau site occupies an area $2\pi\ell_B^2=\frac{2\pi}{B}$.
  - Correlation function between a pair of particles takes the form

$$\Omega(1,2)=|\Psi^{\mathbf{1}}|^2|\Psi^{\mathbf{2}}|^2-|\Psi_\Lambda^{*\mathbf{1}}\Psi_\Lambda^{\mathbf{2}}|^2,$$
$$=1-e^{-2B|\vec{X}^{\mathbf{1}}-\vec{X}^{\mathbf{2}}|^2},$$

  - $\Omega(1,2)$ approaches 1 at separations $\gg \ell_B$. Probability of finding two particles at the same location is zero.

QHE on S^4

- Landau problem for charged particles on S^4 formulated and solved by Hu and Zhang (2000).
- Particles are under influence of a background $SU(2)$ gauge field. This is provided by a Yang monopole.
- Multiparticle problem: In LLL, with filling factor $\nu = 1$, finite spatial density occurs iff the charges particles carry infinitely large IRR's of $SU(2)$.
- In $2D$ edge excitations give spin zero particles (massless chiral bosons), in $4D$ edge excittions have higher spin particles like photons and gravitons. However, other massless higher-spin states also occur.
- Effective Abelian and non-Abelian Chern-Simons theory descriptions in $6+1$ and $4+1$, respectively are also given as generalizations of effective CS theory for QHE in the low energy regime.

Motivations from String Theory

- Low energy dynamics of strings-D-branes configurations are effectively captured by QHE on S^2 and S^4. Bernevig et.al. (2001), Fabinger (2002). Very roughly:
  - Wrap a $D2$-brane on S^2 and dissolve N, $D0$-branes on it.
  - Take K flat $D6$-branes $\perp$ $D2$-brane and move them to the center of $D2$-brane.
  - Due to topological constraints K fundamental strings stretch between $D2$ and $D6$-branes.
  - In the low energy limit, charged string-ends may be viewed as K-charged particles, N is interpreted as the magnetic flux and $\nu = \frac{K}{N}$ as the filling factor.
  - Background magnetic field may be viewed as density of $D0$-branes on the $D2$-brane.
  - $D0$-branes may be viewed to form an incompressible liquid.
- Low energy dynamics of $D4$-$D4$-branes system with strings stretching in between is effectively captured by QHE on S^4.

QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$

- Grassmann Manifold $\mathbf{Gr}_2(\mathbb{C}^4)$ may be realized as the coset

$$\mathbf{Gr}_2(\mathbb{C}^4) \equiv \frac{SU(4)}{SU(2) \times SU(2) \times U(1)} .$$

- We label the left- and right-invariant vector fields of $SU(4)$ as L_α and R_α, $\alpha = 1, \cdots, 15$.
- $L_i^1, R_i^1, L_i^2, R_i^2$ $i = 1,2,3$ $(\alpha = 1, \cdots, 6)$ are the left, right generators of $SU(2) \times SU(2)$ with $L_i^a L_i^a = L(L+1)$, $R_i^a R_i^a = R(R+1)$ $(a = 1, 2$, no sum over $a)$.
- 8 tangent vectors along $\mathbf{Gr}_2(\mathbb{C}^4)$ may be given by R_α, $\alpha = 7, \cdots, 14$.
- Hamiltonian for charged particles on $\mathbf{Gr}_2(\mathbb{C}^4)$

$$H = \frac{1}{2M\ell^2} \sum_{\alpha=7}^{14} R_\alpha^2$$

- Wigner $\mathcal{D}$-functions:

$$\mathcal{D}^{(p,q,r)}_{L^1,L^1_3,L^2,L^2_3,L_{15};R^1,R^1_3,R^2,R^2_3,R_{15}}$$

- Integers (p,q,r) label the irreducible representations of $SU(4)$.
- R_{15} gives the $U(1)$ magnetic charge background.
- R^1 and R^2 give the non-Abelian magnetic charge backgrounds.
- Hamiltonian is expressed as

$$\begin{aligned} H &= \frac{1}{2M\ell^2}\left(\sum_{\alpha=1}^{15} R^2_\alpha - \mathcal{C}^R_2 - R^2_{15}\right) \\ &= \frac{1}{2M\ell^2}\left(\mathcal{C}_2(p,q,r) - \mathcal{C}^R_2 - R^2_{15}\right) \end{aligned}$$

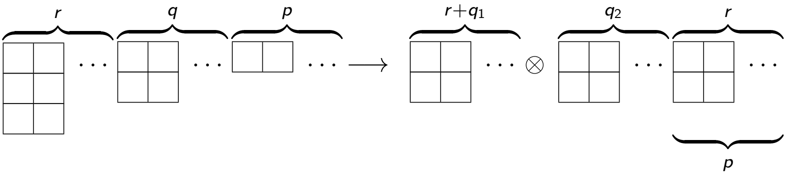
- Quadratic Casimirs of $SU(4)$ and $SU(2)\times SU(2)$ have the eigenvalues

$$\mathcal{C}_2(p,q,r) = \frac{3}{8}(r^2+p^2)+\frac{1}{2}q^2+\frac{1}{8}(2pr+4pq+4qr+12p+16q+12r).$$

$$\mathcal{C}^R_2 = R^1(R^1+1)+R^2(R^2+1)$$

Abelian Background

- Consider the branching of $SU(4)$ IRR $(p, q = q_1 + q_2, r)$ under $SU(2) \times SU(2) \times U(1)$



- We want to restrict to $SU(4)$ Wigner $\mathcal{D}$-functions, transforming as singlet under $SU(2) \times SU(2)$ and carry non-zero $U(1)$-charge. That is,

$$R_i^1 = R_i^2 = 0\,, \quad R_{15} = \frac{1}{\sqrt{2}}(q_1 - q_2) =: \frac{1}{\sqrt{2}}n\,,$$

- Observe that for this to happen $p = r$ and $\mathcal{D}$-functions take the form

$$\mathcal{D}^{(p,2q_1,p)}_{L^1,L^1_3,L^2,L^2_3,L_{15};0,0,0,0,\frac{1}{\sqrt{2}}n}$$

- Spectrum of the Hamiltonian at fixed n is labeled by two integers (p, q_2) and reads, with $\mathcal{C}_2^R = 0$,

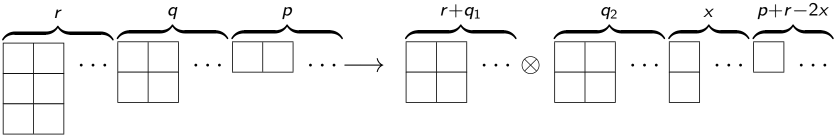
$$
\begin{aligned}
H &= \frac{1}{2M\ell^2}\left(\mathcal{C}_2(p,q,r) - R_{15}^2\right) \\
&= \frac{1}{2M\ell^2}\left(p^2 + 3p + np + 2q_2^2 + 4q_2 + 2pq_2 + 2n(1+q_2)\right) .
\end{aligned}
$$

- Degeneracy of each LL is given by the dimension of the IRR $(p\,,q\,,p)$.
- The LLL is specified by $p = q_2 = 0$ and has the energy $\frac{n}{M\ell^2} = \frac{2B}{M}$. $(B = \frac{n}{2\ell^2})$
- For the many particle problem, if all states of LLL are filled with the filling factor $\nu = 1$, in the thermodynamic limit $\ell \to \infty, \mathcal{N} \to \infty$, spatial density of particles is finite:

$$
\rho = \frac{dim(0\,,n\,,0)}{\frac{\pi^4\ell^8}{12}} \underset{\ell\to\infty\,,\mathcal{N}\to\infty}{\longrightarrow} \frac{n^4}{\pi^4\ell^8} = \left(\frac{2B}{\pi}\right)^4 ,
$$

Single $SU(2)$ and $U(1)$ Gauge Field Background

- In this case, $R^1 = 0$ and R^2 takes a range of spin values labeling the possible non-Abelian $SU(2)$ charges.
- A generic branching may be given as $(0 \leq x \leq r)$



- We have

$$R^1 = 0,\ R^2 = \frac{p-r}{2}, \cdots, \frac{p+r}{2},$$

$$R_{15} = \frac{1}{2\sqrt{2}}(2(q_1 - q_2) - (p-r)) := \frac{n}{\sqrt{2}}$$

- $n \in \mathbb{Z}$ implies $m := \frac{p-r}{2} \in \mathbb{Z}$
- Wave functions are given by

$$\mathcal{D}^{(p,q_1+q_2,r)}_{L^1,L^1_3 L^2,L^2_3 L_{15};0,0,R^2,R^2_3,\frac{n}{\sqrt{2}}}$$

- At fixed R^2 and n, energy eigenvalues are labeled by two integers $(q_2\,, m)$

$$E = \frac{1}{2M\ell^2}\left(2q_2^2+2q_2(n+R^2+m+2)+n(R^2+m+2)+(R^2+m)(2+m)\right).$$

- The LLL is given by setting $q_2 = m = 0$

$$E = \frac{1}{2M\ell^2}(n(R^2+2)+2R^2)$$

- In thermodynamic limit, spatial densities at filling factor $\nu = 1$ are rendered finite:
  i) For pure $SU(2)$ background $n = 0$ and R^2 scales as ℓ^2:

$$\rho \sim \frac{\dim(R^2\,,0\,,R^2)}{\frac{\pi^4\ell^8}{12}(2R^2+1)} \underset{\ell\to\infty\,,\mathcal{N}\to\infty}{\longrightarrow} \frac{\left(R^2\right)^4}{\pi^4\ell^8}$$

  ii) For both $U(1)$ and $SU(2)$ backgrounds, we may have n scale as ℓ^2 and R^2 to be finite,

$$\rho \sim \frac{\dim(R^2\,,n\,,R^2)}{\frac{\pi^4\ell^8}{12}(2R^2+1)} \underset{\ell\to\infty\,,\mathcal{N}\to\infty}{\longrightarrow} \frac{n^4}{2\pi^4\ell^8R^2}\,,$$

$SU(2) \times SU(2)$ *and* $U(1)$ *Gauge Field Backgrounds*

- Without going into details, the wave functions may be given as

$$\mathcal{D}^{(p_1+p_2,q_1+q_2+x,r)}_{L^1,L^1_3 L^2,L^2_3 L_{15};R^1,R^1_3,R^2,R^2_3,\frac{n}{\sqrt{2}}}$$

- R^1 and R^2 take on the values

$$R^1 = \frac{p_1+x}{2}\,, \quad \frac{|2\mathcal{M}-S|}{2} \leq R^2 \leq \frac{S}{2}$$
$$0 \leq x \leq q\,, \quad 0 \leq p_1 \leq p\,, \quad S = p_2 + x + r$$

  $\mathcal{M}$ is the largest among the integers p_2, x and r.
- $U(1)$ charge is (Here either $q_2 = 0$ or $p_1 = 0$)

$$n = \frac{1}{2}\left(2(q_1-q_2) - (p_2 - p_1 - r)\right)$$

- For fixed R^1, R^2 and n, energy spectrum depends on two integers and LLL energy is

$$E = \frac{1}{2M\ell^2}\left(n(R^1+R^2+2) + 2R^2\right)$$

- In the thermodynamic limit, finite spatial densities are achieved in this case too.

QHE on $\mathbf{Gr}_2(\mathbb{C}^N)$

- Coset realization of $\mathbf{Gr}_2(\mathbb{C}^N)$ reads

$$\mathbf{Gr}_2(\mathbb{C}^N) = \frac{SU(N)}{SU(N-2) \times SU(2) \times U(1)},$$

- $SU(N)$ Wigner $\mathcal{D}$-functions,

$$\mathcal{D}^{(P_1,P_2,P_3,\ldots,P_{N-2},P_{N-1})}_{L^{SU(N-2)}\,,L\,,L_3\,,L_{N^2-1}\,,R^{SU(N-2)}\,,R\,,R_3\,,R_{N^2-1}},$$

- IRR's of $SU(N)$ are labeled by $N-1$ integers $P_1, P_2, P_3, \ldots, P_{N-2}, P_{N-1}$.
- Hamiltonian for charged particles on $\mathbf{Gr}_2(\mathbb{C}^N)$ takes the form :

$$H = \frac{1}{2M\ell^2}\left(\mathcal{C}_2^{SU(N)} - \mathcal{C}_2^{SU(N-2)} - \mathcal{C}_2^{SU(2)} - R^2_{N^2-1}\right)$$

- For pure $U(1)$ background gauge field $R^{SU(N-2)} = 0\,, \quad R = 0$, branching of the $SU(N)$ IRR requires

$$(P_1\,, P_2\,, 0\,, \cdots\,, 0\,, P_{N-2}\,, P_1)$$

- $U(1)$ background charge is given as

$$R_{N^2-1} = \sqrt{1 - \frac{2}{N}} n, \quad n = P_{N-2} - P_2$$

- LLL energy reads

$$E = \frac{Nn - 2n}{2M\ell^2} .$$

- For $SU(2)$ and $U(1)$ background gauge fields branching rules restrict to the IRRs $(P_1\,, P_2\,, 0\,, \cdots\,, 0\,, P_{N-2}\,, P_{N-1})$
- $SU(2)$ spin takes values in the range

$$R = \frac{P_1 - P_{N-1}}{2}\,, \cdots\,, \frac{P_{N-1} + P_1}{2}$$

- $U(1)$ charge is

$$R_{N^2-1} = \sqrt{1 - \frac{2}{N}} n, \quad n = \frac{1}{2}\left(P_{N-1} + 2(P_{N-2} - P_2) - P_1\right)$$

- LLL energy is

$$E = \frac{1}{2M\ell^2}\left(nR + (N-2)(n+R)\right) ,$$

Local Form of the Wave Functions on $\mathbf{Gr}_2(\mathbb{C}^4)$

- Plücker embedding $\mathbf{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbf{P}\left(\bigwedge^k \mathbb{C}^n\right)$
- For $\mathbf{Gr}_2(\mathbb{C}^4)$, this embedding is $\mathbf{Gr}_2(\mathbb{C}^4) \hookrightarrow \mathbf{P}\left(\mathbb{C}^4 \wedge \mathbb{C}^4\right) \equiv \mathbb{C}P^5$.
- Introducing one set of complex coordinates for each $\mathbb{C}^4$ as $v_\alpha\,, w_\alpha$ $(\alpha = 1\,, \cdots\,, 4)$, Plücker coordinates are constructed as

$$P_{\alpha\beta} = \frac{1}{\sqrt{2}}(v_\alpha w_\beta - v_\beta w_\alpha)\,.$$

- $P_{\alpha\beta} \sim \lambda P_{\alpha\beta}$ where $\lambda \in U(1)$ and $\sum_{\alpha\,,\beta}^4 |P_{\alpha\beta}|^2 = 1$.
- Plücker embedding is specified by the homogeneous condition

$$\varepsilon_{\alpha\beta\gamma\delta} P_{\alpha\beta} P_{\gamma\delta} = P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0\,,$$

- $P_{\alpha\beta}$ may be used to parametrize the columns of $g \in SU(4)$ in the IRR $(0, 1, 0)$.

$$g := \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \begin{matrix} P_{34}^* & P_{12} \\ -P_{24}^* & P_{13} \\ P_{23}^* & P_{14} \\ P_{14}^* & P_{23} \\ -P_{13}^* & P_{24} \\ P_{12}^* & P_{34} \end{matrix} \end{pmatrix}$$

- Wave functions $\mathcal{D}^{(0\,,q_1+q_2\,,0)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}\,;\,0\,,0\,,0\,,0\,,\frac{n}{\sqrt{2}}}(g)$ fullfill the gauge transformation property

$$\mathcal{D}^{(0\,,q_1+q_2\,,0)}(gh) = \mathcal{D}^{(0\,,q_1+q_2\,,0)}(ge^{i\lambda_{15}\theta}) = \mathcal{D}^{(0\,,q_1+q_2\,,0)}(g)e^{i\frac{n}{\sqrt{2}}\theta}\,.$$

- In the IRR $(0,1,0)$ this implies, using $\lambda_{15} = \frac{1}{\sqrt{2}}\mathrm{diag}(0,0,0,0,-1,1)$

$$\mathcal{D}^{(0\,,1\,,0)}(g) \sim P_{\alpha\beta}\,.$$

- IRR $(0,q,0)$ is the q-fold symmetric tensor product of $(0,1,0)$. (i.e. $(0,q,0) \equiv \prod_{\otimes q}(0,1,0)$). We infer

$$\mathcal{D}^{(0\,,q_1+q_2\,,0)}(g) \sim P_{\alpha_1\beta_1}P_{\alpha_2\beta_2}\cdots P_{\alpha_{q_1}\beta_{q_1}}P^*_{\gamma_1\delta_1}P^*_{\gamma_2\delta_2}\cdots P^*_{\gamma_{q_2}\delta_{q_2}}$$

- LLL wave functions are holomorphic in Plücker coordiantes ($q_2 = 0$),

$$\mathcal{D}^{(0\,,q_1\,,0)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}\,;\,0\,,0\,,0\,,0\,,\frac{n}{\sqrt{2}}}(g) \sim P_{\alpha_1\beta_1}P_{\alpha_2\beta_2}\cdots P_{\alpha_{q_1}\beta_{q_1}}\,,$$

Correlation Function and Incompressibility

- Multiparticle wave function is given by the Slater determinant

$$\Psi_{MP} = \frac{1}{\sqrt{\mathcal{N}!}} \varepsilon^{\Lambda_1 \Lambda_2 \cdots \Lambda_n} \Psi_{\Lambda_1}(P^{(1)}) \Psi_{\Lambda_2}(P^{(2)}) \cdots \Psi_{\Lambda_N}(P^{(N)})$$

- For $\Psi_{\Lambda_i}(P^{\mathbf{i}}) \equiv \Psi^{\mathbf{i}}_{\Lambda_i} \sim (P^{\mathbf{i}}_{\alpha\beta})^n$, correlation function between a pair of particles gives

$$\begin{aligned} \Omega(1,2) &= |\Psi^{\mathbf{1}}|^2 |\Psi^{\mathbf{2}}|^2 - |\Psi^{*\mathbf{1}}_{\Lambda} \Psi^{\mathbf{2}}_{\Lambda}|^2, \\ &\xrightarrow[\ell \to \infty, \mathcal{N} \to \infty]{} 1 - e^{-2B\left|\vec{X}^{\mathbf{1}} - \vec{X}^{\mathbf{2}}\right|^2}, \end{aligned}$$

- Probability of finding two particles at the same point approaches to 0. Signalling incompressibility of the Hall fluid.

Local Form of the Abelian Gauge Field

- $U(1)$ gauge potential is

$$A = -\frac{in}{\sqrt{2}} \mathrm{Tr} \left(\lambda_{(6)}^{15} g^{-1} dg \right) = -inP_N^* dP_N$$

with the gauge transformation property $A \longrightarrow A + d(\frac{n\theta}{\sqrt{2}})$.

- Corresponding field strength

$$F = dA = -\frac{in}{\sqrt{2}} \mathrm{Tr} \left(\lambda_{(6)}^{15} g^{-1} dg \wedge g^{-1} dg \right) = -indP_N^* \wedge dP_N \,.$$

- F is antisymmetric, gauge invariant, closed 2-form on $\mathbf{Gr}_2(\mathbb{C}^4)$. As such it is proportional to the Kähler two-form Ω over $\mathbf{Gr}_2(\mathbb{C}^4)$: $F = n\Omega$.
- $\frac{1}{2\pi} \int_\Sigma F = n$, where Σ is a noncontractable two surface in $\mathbf{Gr}_2(\mathbb{C}^4)$. This is the analog of Dirac quantization condition in the present context.

Remarks and Conclusions

- We have studied QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ gave the energy spectrum and wave functions for both Abelian and non-Abelian background gauge fields. Generalized to QHE on $\mathbf{Gr}_2(\mathbb{C}^N)$.
- Relation to fuzzy spaces may be investigated.
- It is known that (Nair and Karabali (2002), Bernevig et. al. (2003))
  - $U(1)$ QHE on $\mathbb{C}P^3 \equiv SU(2)$ gauge field QHE on S^4,
  - $U(1)$ QHE on $\mathbb{C}P^7 \equiv SO(8)$ gauge field QHE on S^8,

- $St_2(\mathbb{R}^6) = \frac{Spin(6)}{Spin(4)}$
- We have the fibration $U(1) \to St_2(\mathbb{R}^6) \to \mathrm{Gr}_2(\mathbb{C}^4)$
- We also have another fibration $S^4 \to St_2(\mathbb{R}^6) \to S^6$.
- Locally we have $\mathbf{Gr}_2(\mathbb{C}^4) \approx \frac{S^5 \times S^4}{U(1)}$
- We conjecture that $U(1)$ QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ is equivalent to QHE on S^5 with S^4 fibers.
- S^4 fibers are associated to an $SO(5)$ gauge field.
- It is known that QHE on $S^3 = \frac{Spin(4)}{Spin(3)}$(Nair and Randjbar-Daemi (2004)) takes the constant background gauge field as the spin connection.
- It is natural to expect that for QHE on $S^5 = \frac{Spin(6)}{Spin(5)}$ one should take $SO(5)$ background gauge field as the spin connection.