The Fischer decomposition of polynomials on superspace

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(joint work with D. Šmíd and V. Souček)

Group30, Ghent 2014

Harmonic analysis on Euclidean space

Notation:

- $\mathcal{P}_k$ ... k-homogeneous polynomials in $\mathbb{R}^m$
- $\mathcal{H}_k = \{P \in \mathcal{P}_k |\ \Delta P = 0\}$ where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_m}^2$

The Fischer decomposition

$$\mathcal{P}_k = \mathcal{H}_k \oplus R^2 \mathcal{P}_{k-2} = \mathcal{H}_k \oplus R^2 \mathcal{H}_{k-2} \oplus R^4 \mathcal{H}_{k-4} \oplus \cdots$$

where $R^2 = x_1^2 + \cdots + x_m^2$.

Representation theory: Irreducible decomposition of $\mathcal{P}_k$ w.r.t. $so(m)$

Harmonic analysis on superspace

F. Sommen, H. De Bie, D. Eelbode, K. Coulembier, R. Zhang and others.

Polynomials on superspace $\mathbb{R}^{m|2n}$

Let $\mathcal{P}$ be the algebra generated by $x_1, x_2, \dots, x_m$, $\grave{x}_1, \grave{x}_2, \dots, \grave{x}_{2n}$ such that

$$x_i x_j = x_j x_i, \quad \grave{x}_i \grave{x}_j = -\grave{x}_j \grave{x}_i, \quad x_i \grave{x}_j = \grave{x}_j x_i$$

We have $\mathcal{P} = \mathbb{R}[x_1, x_2, \dots, x_m] \otimes \Lambda_{2n}$
where Λ_{2n} is the Grassman algebra generated by $\grave{x}_1, \grave{x}_2, \dots, \grave{x}_{2n}$.

Question

Is there the Fischer decomposition of the space $\mathcal{P}_k$ of k-homogeneous polynomials on $\mathbb{R}^{m|2n}$?

YES, if the superdimension $M = m - 2n \notin -2\mathbb{N}_0$ or $m = 0$.

The case $M = m - 2n \notin -2\mathbb{N}_0$

Let $\mathcal{H}_k = \{P \in \mathcal{P}_k |\ \Delta P = 0\}$ where $\Delta = \sum_{j=1}^m \partial_{x_j}^2 - 4\sum_{j=1}^n \partial_{\grave{x}_{2j-1}} \partial_{\grave{x}_{2j}}$

The Fischer decomposition (F. Sommen)

$$\mathcal{P}_k = \mathcal{H}_k \oplus R^2\mathcal{P}_{k-2} = \mathcal{H}_k \oplus R^2\mathcal{H}_{k-2} \oplus R^4\mathcal{H}_{k-4} \oplus \cdots$$

where $R^2 = \sum_{j=1}^m {x_j}^2 - \sum_{j=1}^n \grave{x}_{2j-1}\grave{x}_{2j}$.

This is an irreducible decomposition of $\mathcal{P}_k$ w.r.t. $osp(m|2n)$.

The case $M = m - 2n \in -2\mathbb{N}_0$, $m \neq 0$

Let $I_M = \{k \in \mathbb{N}_0 |\ 2 - M/2 \leq k \leq 2 - M\}$.

(i) Let $k \in I_M$. Then

$$R^{2k+M-2}\mathcal{H}_{2-M-k} \subset \mathcal{H}_k$$

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Theorem (K. Coulembier)

Denote $\mathcal{H}_k^0 = R^{2k+M-2}\mathcal{H}_{2-M-k}$. Under the action of $osp(m|2n)$, $\mathcal{H}_k$ is indecomposable and $\mathcal{H}_k^0$ and $\mathcal{H}_k/\mathcal{H}_k^0$ are irreducible.

A module H is **irreducible** if it has not non-trivial submodules.

A module H is **indecomposable** if it has not non-trivial submodules U, V such that $H = U \oplus V$.

(ii) Let $k \notin I_M$. Then $\mathcal{H}_k$ is irreducible.

Theorem (R.L., V. Souček, D. Šmíd)

We have that $\mathcal{P}_k = \tilde{\mathcal{H}}_k \oplus R^2\Delta R^2\, \mathcal{P}_{k-2}$ *where* $\tilde{\mathcal{H}}_k = \mathrm{Ker}(\Delta R^2 \Delta) \cap \mathcal{P}_k$.

(i) Let $k \notin I_M$. Then $\tilde{\mathcal{H}}_k = \mathcal{H}_k \simeq L^{m|2n}_{(k,0\dots,0)}$ and $\Delta R^2\, \mathcal{P}_{k-2} = \mathcal{P}_{k-2}$.

Here $L^{m|2n}_\lambda$ is an irreducible module with the h.w. λ under $osp(m|2n)$ (following R. Zhang, not V. Kac).

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Here $L^{m|2n}_\lambda$ is an irreducible module with the h.w. λ under $osp(m|2n)$ (following R. Zhang, not V. Kac).

(ii) Let $k \in I_M = \{k \in \mathbb{N}_0 |\ 2 - M/2 \leq k \leq 2 - M\}$.

Theorem (R.L., V. Souček, D. Šmíd)

Under the action of $osp(m|2n)$, $\tilde{\mathcal{H}}_k$ *is indecomposable and*

$$\mathcal{H}^0_k \subset \mathcal{H}_k \subset \tilde{\mathcal{H}}_k \text{ where } \mathcal{H}^0_k = R^{2k+M-2}\mathcal{H}_{2-M-k}.$$

Moreover, we have that

$$\mathcal{H}^0_k \simeq L^{m|2n}_{(2-M-k,0\ldots,0)},\quad \mathcal{H}_k/\mathcal{H}^0_k \simeq L^{m|2n}_{(k,0\ldots,0)},\quad \tilde{\mathcal{H}}_k/\mathcal{H}_k \simeq L^{m|2n}_{(2-M-k,0\ldots,0)}$$

Example: $M = m - 2n \notin -2\mathbb{N}_0$

$$\vdots$$

$$\mathcal{P}_4 = \mathcal{H}_4 \oplus R^2\mathcal{H}_2 \oplus R^4\mathcal{H}_0$$

$$\mathcal{P}_5 = \mathcal{H}_5 \oplus R^2\mathcal{H}_3 \oplus R^4\mathcal{H}_1$$

$$\mathcal{P}_6 = \mathcal{H}_6 \oplus R^2\mathcal{H}_4 \oplus R^4\mathcal{H}_2 \oplus R^6\mathcal{H}_0$$

$$\mathcal{P}_7 = \mathcal{H}_7 \oplus R^2\mathcal{H}_5 \oplus R^4\mathcal{H}_3 \oplus R^6\mathcal{H}_1$$

$$\mathcal{P}_8 = \mathcal{H}_8 \oplus R^2\mathcal{H}_6 \oplus R^4\mathcal{H}_4 \oplus R^6\mathcal{H}_2 \oplus R^8\mathcal{H}_0$$

$$\mathcal{P}_9 = \mathcal{H}_9 \oplus R^2\mathcal{H}_7 \oplus R^4\mathcal{H}_5 \oplus R^6\mathcal{H}_3 \oplus R^8\mathcal{H}_1$$

$$\vdots$$

Example: $M = -6,\ m \neq 0$

Here $I_M = \{5, 6, 7, 8\}$ and, for $k \in I_M$,

$$\mathcal{H}_k^0 \subset \mathcal{H}_k \subset \tilde{\mathcal{H}}_k$$

where $\mathcal{H}_5^0 = R^2\mathcal{H}_3$, $\mathcal{H}_6^0 = R^4\mathcal{H}_2$, $\mathcal{H}_7^0 = R^6\mathcal{H}_1$, $\mathcal{H}_8^0 = R^8\mathcal{H}_0$.

$$\begin{aligned}
&\vdots\\
\mathcal{P}_4 &= \mathcal{H}_4 \oplus R^2\mathcal{H}_2 \oplus R^4\mathcal{H}_0\\
\mathcal{P}_5 &= \tilde{\mathcal{H}}_5 \oplus R^4\mathcal{H}_1\\
\mathcal{P}_6 &= \tilde{\mathcal{H}}_6 \oplus R^2\mathcal{H}_4 \oplus R^6\mathcal{H}_0\\
\mathcal{P}_7 &= \tilde{\mathcal{H}}_7 \oplus R^2\tilde{\mathcal{H}}_5\\
\mathcal{P}_8 &= \tilde{\mathcal{H}}_8 \oplus R^2\tilde{\mathcal{H}}_6 \oplus R^4\mathcal{H}_4\\
\mathcal{P}_9 &= \mathcal{H}_9 \oplus R^2\tilde{\mathcal{H}}_7 \oplus R^4\tilde{\mathcal{H}}_5\\
&\vdots
\end{aligned}$$

The Fischer decomposition

Let $I_M = \{k \in \mathbb{N}_0 | \ 2 - M/2 \leq k \leq 2 - M\}$.

Theorem (R.L., V. Souček, D. Šmíd)

Let $J_k = \{k - 2j | \ j = 0, \dots, \lfloor k/2 \rfloor\}$ *and* $I^0_{M,k} = \{2 - M - \ell | \ \ell \in J_k \cap I_M\}$*. We have that*

$$\mathcal{P}_k = \bigoplus \{R^{k-\ell}\tilde{\mathcal{H}}_\ell | \ \ell \in J_k \cap I_M\} \oplus \bigoplus \{R^{k-\ell}\mathcal{H}_\ell | \ \ell \in J_k \setminus (I_M \cup I^0_{M,k})\}$$

This is a decomposition into indecomposable submodules w.r.t. $osp(m|2n)$*.*

Next step: Analogous results for spinor valued sph. monogenics on $\mathbb{R}^{m|2n}$

K. Coulembier, H. De Bie: super spinor space, super Dirac operator, the Fischer decomposition only when $M = m - 2n \notin -2\mathbb{N}_0$