On the boundaries of quantum integrability for the spin-1/2 Richardson–Gaudin system

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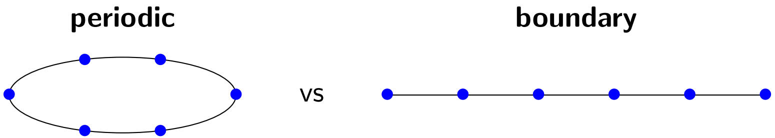
Some History

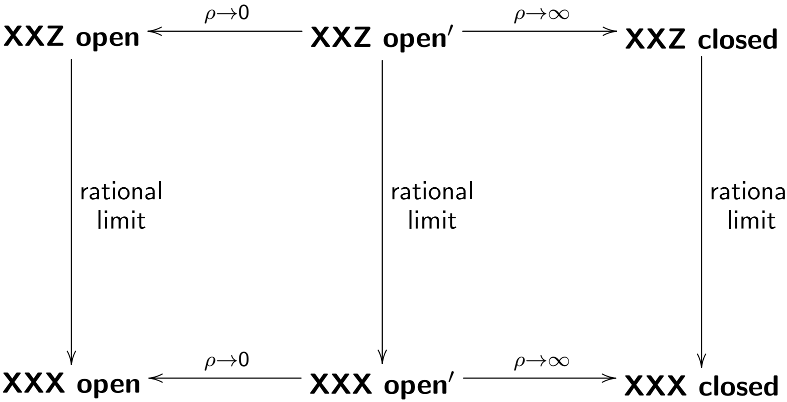
- **Quantum Inverse Scattering Method (QISM)** for twisted periodic boundary conditions [Faddeev, Kulish, Sklyanin, Takhtadzhan 1979].
- **Boundary QISM (BQISM)** for open-boundary conditions, for the case of the Heisenberg XXZ spin chain [Sklyanin 1988].



- What is the effect of the "boundary" for **Richardson-Gaudin models**?
  - **Gaudin magnet** with boundary [Hikami 1995].
  - **Quasi-classical limit** of the BQISM [Di Lorenzo et al. 2002].
  - **Generalized** Gaudin systems [Skrypnyk 2006, 2007, 2010].
  - **Trigonometric** Gaudin model [Cirilio António et al. 2013].

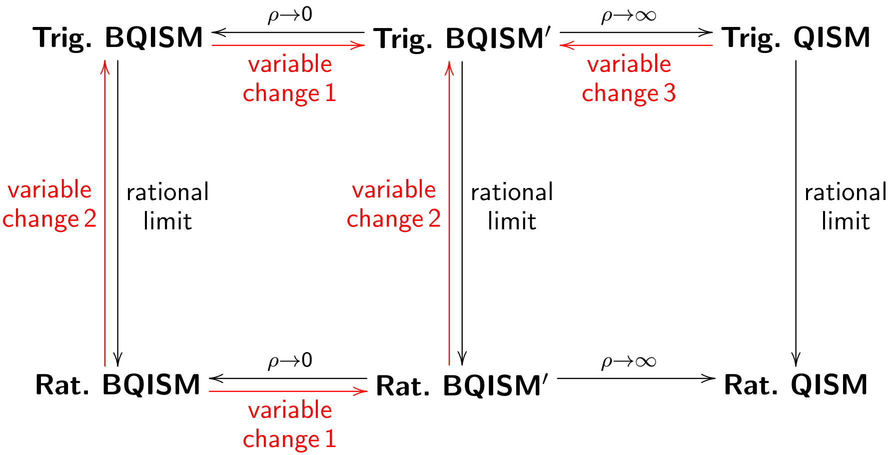
Outline

For the **Heisenberg model**:



Outline

For the **Richardson-Gaudin models**:



R-matrix

R-**matrix** is an operator $R(u) \in \mathrm{End}(V \otimes V)$ $(V = \mathbb{C}^2,\ u \in \mathbb{C})$ satisfying the **Yang-Baxter Equation (YBE)** in $\mathrm{End}(V \otimes V \otimes V)$:

$$\boxed{R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)}$$

- **Rational solution** $(\eta \in \mathbb{C},\quad P(u \otimes v) = v \otimes u,\ \forall u, v \in V)$

$$R^{rat}(u) = \frac{1}{u+\eta}(uI \otimes I + \eta P) = \frac{1}{u+\eta}\begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}.$$

- **Trigonometric solution**

$$R^{trig}(u) = \frac{1}{\sinh(u+\eta)}\begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}.$$

Rational limit $\lim_{\nu\to 0} \dfrac{\sinh(\nu x)}{\nu} = x$: **Trigonometric $\longrightarrow$ Rational**

QISM [Faddeev et al. 1979]

Monodromy matrix $\in \text{End}\left(V_a \otimes V^{\otimes \mathcal{L}}\right)$ (where $V_a = \mathbb{C}^2$ is the auxiliary space, $V^{\otimes \mathcal{L}} = \underbrace{V \otimes V \otimes \ldots \otimes V}_{\mathcal{L} \text{ times}}$ is the quantum space, $\mathcal{L} \in \mathbb{N}$):

$$T_a(u) = \begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix}_a R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\ldots R_{a2}(u-\varepsilon_2)R_{a1}(u-\varepsilon_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a .$$

Transfer matrix $t(u) = tr_a(T_a(u)) = A(u) + D(u) \in \text{End}(V^{\otimes \mathcal{L}})$:

$$\boxed{[t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C}}$$

Then $t(u)$ generates a set of **mutually commuting** operators $\{C_j\}$:

$$t(u) = \sum_{j=-\infty}^{\infty} C_j u^j .$$

Take any function of $\{C_j\}$ as the Hamiltonian $\Rightarrow$ **integrals of motion**.

Bethe Ansatz Equations

Algebraic Bethe Ansatz: start with the **reference state** $\Omega = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes \mathcal{L}}$

$$B(u)\Omega = 0, \; A(u)\Omega = a(u)\Omega, \; D(u)\Omega = d(u)\Omega, \; C(u)\Omega \neq 0,$$

and look for other eigenstates of $t(u) = A(u) + D(u)$ in the form

$$\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega.$$

Parameters v_k must satisfy the **Bethe Ansatz Equations (BAE)**:

$$e^{-2\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)} = \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}, \; k = 1, ..., N$$

Take first non-zero term as $\eta \to 0$ to obtain the **quasi-classical limit** of the QISM $\Rightarrow$ **Richardson-Gaudin models**.

Richardson-Gaudin models (quasi-classical limit)

- **Trig. QISM** [Amico et al. 2001], [Dukelsky et al. 2001]

$$\boxed{2\gamma + \sum_{l=1}^{\mathcal{L}} \coth(v_k - \varepsilon_l) = 2\sum_{i\neq k}^{N} \coth(v_k - v_i)} \qquad (1)$$

- **Rat. QISM** [Richardson 1963] (using $\nu \coth(\nu x) \to \frac{1}{x}$ as $\nu \to 0$)

$$\boxed{2\gamma' + \sum_{l=1}^{\mathcal{L}} \frac{1}{v_k - \varepsilon_l} = \sum_{i\neq k}^{N} \frac{2}{v_k - v_i}} \qquad (0)$$

Reflection equations [Cherednik 1984]

Start with the **trigonometric** R-matrix. In addition to the YBE we require that it satisfies the **reflection equations** for some $K^{\pm} \in \mathrm{End}(V)$, referred to as the **reflection matrices**:

$$\begin{cases} R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \\ R_{12}(v-u)K_1^+(u)\mathcal{R}_{21}(u+v)K_2^+(v) = K_2^+(v)\mathcal{R}_{12}(u+v)K_1^+(u)R_{21}(v-u), \end{cases}$$

where $\mathcal{R}(u) = R(-u-2\eta)$. Easy to check that

$$K^-(u) = \begin{pmatrix} \sinh(\xi^- + u) & 0 \\ 0 & \sinh(\xi^- - u) \end{pmatrix},$$

$$K^+(u) = \begin{pmatrix} \sinh(\xi^+ + u + \eta) & 0 \\ 0 & \sinh(\xi^+ - u - \eta) \end{pmatrix}$$

satisfy these equations for any $\xi^{\pm} \in \mathbb{C}$.

BQISM [Sklyanin 1988]

Define the **double row monodromy matrix** $\in \mathrm{End}\left(V_a \otimes V^{\otimes \mathcal{L}}\right)$:

$$T_a(u) = R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})...R_{a1}(u-\varepsilon_1)K_a^-(u)\times$$
$$\times\ R_{a1}(u+\varepsilon_1)...R_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}}).$$

The **transfer matrix**

$$t(u) = tr_a\left(K_a^+(u)T_a(u)\right)$$

again satisfies

$$\boxed{[t(u), t(v)] = 0 \ \ \forall u, v \in \mathbb{C}}$$

Thus, it is a generating function for the **integrals of motion**!

Unified Approach

Introduce $\rho \in \mathbb{C}$ by **variable change 1**: $u \to u + \rho/2, \ \varepsilon_l \to \varepsilon_l + \rho/2$.
Modified **transfer matrix**:

$$t(u) = \ tr_a\Big(K_a^+(u+\rho/2)R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})...R_{a1}(u-\varepsilon_1)\times$$
$$\times K_a^-(u+\rho/2)R_{a1}(u+\varepsilon_1+\rho)...R_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}}+\rho)\Big).$$

- $\rho \to 0$ yields Sklyanin's formulation,
- $\rho \to \infty$ ("**attenuated limit**") yields twisted-periodic QISM:

$$\boxed{t(u) \xrightarrow{\rho\to\infty} tr_a\left(\begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma}\end{pmatrix}_a R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})...R_{a1}(u-\varepsilon_1)\right)}$$

where

$$\gamma = \mathcal{L}/2 - N - \eta^{-1}(\xi^+ + \xi^-), \ \hat{N} = \sum_{j=1}^{\mathcal{L}} \hat{N}_j, \quad \hat{N}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_j.$$

Bethe Ansatz Equations

Set $\rho = 0$ for the moment. The **BAE** in this case:

$$\boxed{\begin{aligned} &\frac{\sinh(\xi^+ + v_k + \eta/2)}{\sinh(\xi^+ - v_k + \eta/2)} \frac{\sinh(\xi^- + v_k + \eta/2)}{\sinh(\xi^- - v_k + \eta/2)} \times \\ &\times \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2) \sinh(v_k + \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2) \sinh(v_k + \varepsilon_l + \eta/2)} = \\ &= \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta) \sinh(v_k + v_i - \eta)}{\sinh(v_k - v_i + \eta) \sinh(v_k + v_i + \eta)} \end{aligned}}$$

If we substitute $\eta = 0$ the BAE will take the following form:

$$\frac{\sinh(\xi^+ + v_k) \sinh(\xi^- + v_k)}{\sinh(\xi^+ - v_k) \sinh(\xi^- - v_k)} = 1.$$

Assume $\xi^+ = \xi^+(\eta)$, $\xi^- = \xi^-(\eta)$, so that this holds as $\eta \to 0$:

$$\boxed{\xi^+ = \xi + \eta\alpha, \quad \xi^- = -\xi + \eta\beta}$$

Richardson-Gaudin model with boundary

- **Trig. BQISM** (denote $\delta = -(\alpha + \beta + 1)$) (2)

$$\delta\left(\coth(v_k - \xi) + \coth(v_k + \xi)\right) + \sum_{l=1}^{\mathcal{L}} \left(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l)\right) = \\ = 2\sum_{i \neq k}^{N} \left(\coth(v_k - v_i) + \coth(v_k + v_i)\right)$$

- **Trig. BQISM′** is obtained by $v_k \mapsto v_k + \rho/2$, $\varepsilon_l \mapsto \varepsilon_l + \rho/2$:

$$\delta\left(\coth(v_k + \rho/2 - \xi) + \coth(v_k + \rho/2 + \xi)\right) + \\ + \sum_{l=1}^{\mathcal{L}} \left(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l + \rho)\right) = \\ = 2\sum_{i \neq k}^{N} \left(\coth(v_k - v_i) + \coth(v_k + v_i + \rho)\right) \qquad (2')$$

Attenuated limit: $\rho \to \infty$

Note that $\coth x \to 1$ as $x \to \infty$.

Thus, as $\rho \to \infty$ **Trig. BQISM**$'$ ($2'$) tends to

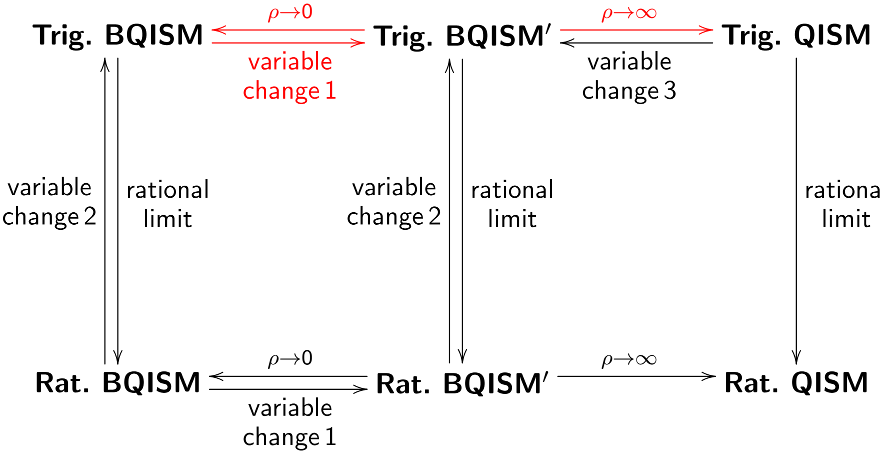
$$2\delta + \sum_{l=1}^{\mathcal{L}} (\coth(v_k - \varepsilon_l) + 1) = 2 \sum_{i \neq k}^{N} (\coth(v_k - v_i) + 1)\,.$$

One can see that it is equivalent to **Trig. QISM** (1):

$$2\gamma + \sum_{l=1}^{\mathcal{L}} \coth(v_k - \varepsilon_l) = 2 \sum_{i \neq k}^{N} \coth(v_k - v_i),$$

where $\gamma = \delta + \mathcal{L}/2 - (N-1)$.

Diagram


Trig. BQISM
ρ→0
variable change 1
Trig. BQISM′
ρ→∞
variable change 3
Trig. QISM
variable change 2
rational limit
variable change 2
rational limit
rational limit
Rat. BQISM
ρ→0
variable change 1
Rat. BQISM′
ρ→∞
Rat. QISM

Rational limit

- **Rat. BQISM** (obtained by the rational limit from **Trig. BQISM** (2))

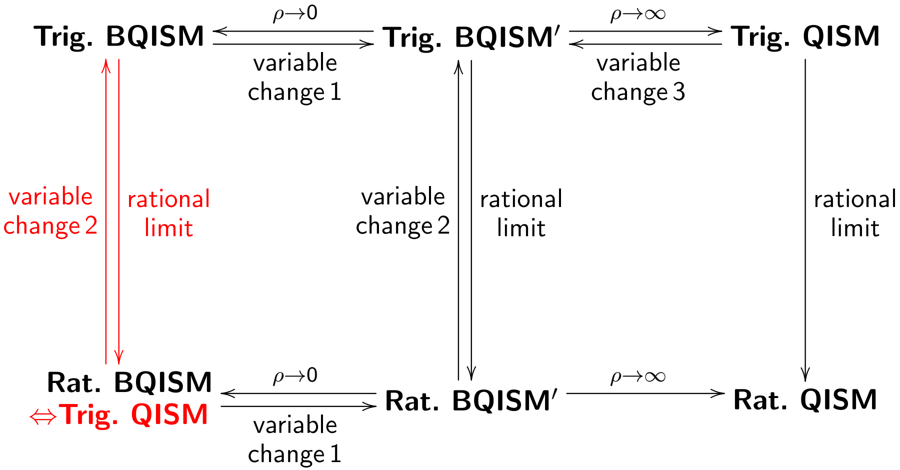
$$\boxed{\frac{\delta}{v_k^2-\xi^2}+\sum_{l=1}^{\mathcal{L}}\frac{1}{v_k^2-\varepsilon_l^2}=2\sum_{i\neq k}^{N}\frac{1}{v_k^2-v_i^2}} \tag{3}$$

One can show that $\boxed{\textbf{Rat. BQISM } (3) \Longleftrightarrow \textbf{Trig. QISM } (1)}$ via an invertible variable change:

$$v_k \mapsto \sqrt{\exp(2v_k)+\xi^2},\ \varepsilon_l \mapsto \sqrt{\exp(2\varepsilon_l)+\xi^2}.$$

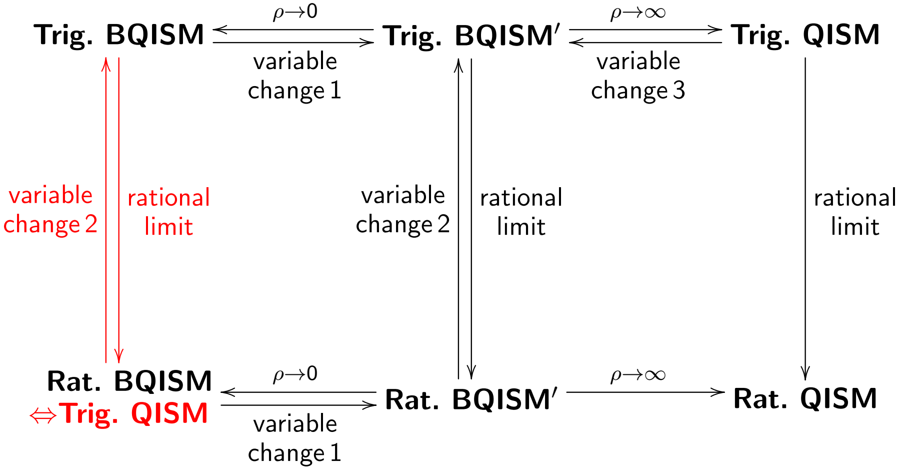
Furthermore, $\boxed{\textbf{Rat. BQISM } (3) \longrightarrow \textbf{Trig. BQISM } (2)}$ by the **variable change 2**:

$$v_k \mapsto \sinh v_k,\ \varepsilon_l \mapsto \sinh \varepsilon_l,\ \xi \mapsto \sinh \xi.$$



For more details see arXiv:1405.7451

Thank you for your attention!



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