Reflection positivity and unitary representations of Lie groups (jt. with Gestur Ólafsson, Stéphane Merigon, Palle Jorgensen)

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Outline

Introduction

Osterwalder and Schrader (1973/75): correspondence between euclidean field theories with symmetry group $\mathrm{Mot}(\mathbb{R}^d) = \mathbb{R}^d \rtimes \mathrm{O}_d(\mathbb{R})$ (euclidean motion group) and relativistic field theories with symmetry group $\mathbb{R}^d \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})_+ = \mathbb{R}^d \rtimes L^\uparrow$ (the orthochronous Poincaré group).

Main point: More accessible Hilbert space $\mathcal{E}$ in euclidean context and commutative field algebra:

$$\mathcal{E} \cong L^2(\mathcal{D}'(\mathbb{R}^d), \mu), \quad \mu \text{ motion invariant measure on } \mathcal{D}'(\mathbb{R}^d),$$

which is reflection positive. OS-quantization leads to the quantum Hilbert space $\widehat{\mathcal{E}}$ with a unitary representation of the Poincaré group and on which the "quantum fields" live.

Goal: Understand OS-quantization (correspondence between euclidean and Poincaré group) for more general Lie groups.

OS-quantization

Definition

A reflection positive Hilbert space is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where $\mathcal{E}$ is a Hilbert space, θ a unitary involution and the subspace $\mathcal{E}_+$ is θ-positive, i.e., $\langle v, v\rangle_\theta := \langle \theta v, v\rangle \geq 0$ for $v \in \mathcal{E}_+$.

We write $\mathcal{N} := \{v \in \mathcal{E}_+ \colon \langle \theta v, v\rangle = 0\}$, $\widehat{\mathcal{E}}$ for the completion of $\mathcal{E}_+/\mathcal{N}$ and $q(v) := \widehat{v} := v + \mathcal{N}$ for the quotient map $q\colon \mathcal{E}_+ \to \widehat{\mathcal{E}}$.
For an (unbounded) operator $T\colon \mathcal{D} \to \mathcal{E}_+$, $\mathcal{D} \subseteq \mathcal{E}_+$, with $T(\mathcal{N}) \subseteq \mathcal{N}$, the operator

$$\widehat{T}\colon \widehat{\mathcal{D}} = \{\widehat{v}\colon v \in \mathcal{D}\} \to \widehat{\mathcal{E}}, \quad \widehat{T}\widehat{v} = \widehat{Tv}$$

is called the OS-quantization of T.

Lemma

(a) $U\mathcal{E}_+ = \mathcal{E}_+$, U *unitary and* $\theta U \theta = U \Rightarrow \widehat{U}$ *unitary.*
(b) $U\mathcal{E}_+ \subseteq \mathcal{E}_+$, U *unitary and* $\theta U \theta = U^{-1} \Rightarrow \widehat{U}^* = \widehat{U}$, $\|\widehat{U}\| \leq 1$.

Reflection positive one-parameter groups

Defn.: A unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$ on $(\mathcal{E},\mathcal{E}_+,\theta)$ is called reflection positive if

$$\theta U_t\theta = U_{-t} \quad \text{for} \quad t\in\mathbb{R}, \qquad \text{and} \qquad U_t\mathcal{E}_+ \subseteq \mathcal{E}_+ \quad \text{for} \quad t>0.$$

Remark: Then the OS-quantization $(\widehat{U}_t)_{t\geq 0}$ is a strongly cont. hermitian contraction semigroup on $\widehat{\mathcal{E}}$. In particular $\widehat{U}_t = e^{-tA}$ for $A=A^*\geq 0$, and $U_t^c := e^{itA}$ is called the dual one-parameter group. This is a passage from $\mathbb{R}$ to $i\mathbb{R}$ (c-duality $\mathbb{R}\to i\mathbb{R}$, "Wick rotation").

Example (The euclidean QFT situation)

$\mathcal{E}=L^2(\mathcal{D}'(\mathbb{R}^d),\mu)$, θ induced by $(t,\mathbf{x})\mapsto(-t,\mathbf{x})$ (time reflection)
μ a probability measure on $\mathcal{D}'(\mathbb{R}^d)$
$\mathcal{E}_+ = \overline{\mathrm{span}\{e^{i\varphi^*}\colon \mathrm{supp}(\varphi)\subseteq \mathbb{R}^d_+ := \mathbb{R}_+\times\mathbb{R}^{d-1}\}}$, where $\varphi^*(D):=D(\varphi)$.
μ reflection positive means that $\mathcal{E}_+$ is θ-positive
U_t induced by time translations $(s,\mathbf{x})\mapsto(t+s,\mathbf{x})$.

Proposition

Let $(U_t)_{t\in\mathbb{R}}$ be reflection positive on $(\mathcal{E},\mathcal{E}_+,\theta)$ with $\mathcal{E}_+$ U-cyclic in $\mathcal{E}$.

(a) OS-quantization commutes with reduction:

$q\colon \mathcal{E}_+ \to \widehat{\mathcal{E}}$ *maps the fixed space* $\mathcal{E}^U$ *isometrically onto* $\widehat{\mathcal{E}}^{\widehat{U}}$.

(b) $\mathcal{E}_\infty := \bigcap_{t>0} U_t\mathcal{E}_+$ *only contributes fixed points:* $q(\mathcal{E}_\infty) \subseteq \widehat{\mathcal{E}}^{\widehat{U}}$.

Reduction to: $\mathcal{E}^U \cong \widehat{\mathcal{E}}^{\widehat{U}} = \{0\}$ and $\mathcal{E}_\infty = \{0\}$.

Theorem (Lax–Phillips representation theorem)

If $(U_t)_{t\in\mathbb{R}}$ is a unitary one-parameter group on $\mathcal{E}$, $\mathcal{E}_+ \subseteq \mathcal{E}$ closed invariant under $(U_t)_{t>0}$, such that $\mathcal{E} = \overline{\bigcup_{t<0} U_t\mathcal{E}_+}$ and $\mathcal{E}_\infty = \bigcap_{t>0} U_t\mathcal{E}_+ = \{0\}$, then $(U,\mathcal{E},\mathcal{E}_+)$ is unitarily equivalent to $(V, L^2(\mathbb{R},\mathcal{K}), L^2(\mathbb{R}_+,\mathcal{K}))$ with a Hilbert space $\mathcal{K}$ and $(V_tf)(x) = f(x-t)$.

$\Rightarrow$ If $(U_t)_{t\in\mathbb{R}}$ is reflection positive with $\mathcal{E}_+$ cyclic and $\mathcal{E}_\infty = \{0\}$, then U has uniform absolutely continuous spectrum (all of $\mathbb{R}$).

Problem: The Lax–Phillips Theorem “does not see θ”.

Definition (Positive definite kernels and functions)

(a) Let X be a set and V be a Hilbert space.

A kernel $Q\colon X \times X \to B(V)$ is called positive definite if

$$\sum_{j,k=1}^{n} \langle Q(x_j, x_k)v_k, v_j\rangle \geq 0 \quad \text{for} \quad x_1, \ldots, x_n \in X, v_1, \ldots, v_n \in V.$$

(b) If G is a group, then $\varphi\colon G \to B(V)$ is called positive definite if $Q(g,h) := \varphi(gh^{-1})$ is a positive definite kernel.

GNS construction (Gelfand/Naimark/Segal): A function $\varphi\colon G \to B(V)$ with $\varphi(\mathbf{1}) = \mathbf{1}_V$ is positive definite if and only if there exists a unitary representation $(\pi, \mathcal{H})$ on a Hilbert space with cyclic subspace V such that

$$\varphi\colon G \to B(V), \quad \varphi(g) := P\pi(g)P^*$$

holds for the orthogonal projection $P\colon \mathcal{H} \to V$.

Then π is called a dilation of φ. It is uniquely determined by φ.

Scalar case: $\varphi(g) = \langle \pi(g)v, v\rangle$ for $v \in \mathcal{H}$.

Theorem (Classification of reflection positive functions on $\mathbb{R}$)

Suppose $(U_t)_{t\in\mathbb{R}}$ is reflection positive on $(\mathcal{E}, \mathcal{E}_+, \theta)$. If $\mathcal{E}_0 \subseteq (\mathcal{E}_+)^\theta$ is $\mathbb{R}$-cyclic in $\mathcal{E}$ with projection $P\colon \mathcal{E} \to \mathcal{E}_0$, then $\varphi(t) := PU_tP^*$ is positive definite on $\mathbb{R}$ and determines $(U, \mathcal{E})$ by GNS construction. There exists a $\mathrm{Herm}_+(\mathcal{E}_0)$-valued measure Q on $[0, \infty[$ with

$$\varphi(t) = \int_0^\infty e^{-\lambda|t|}\, dQ(\lambda), \quad \widehat{\mathcal{E}} \cong L^2([0,\infty[, Q; \mathcal{E}_0), \quad (\widehat{U}_t f)(\lambda) = e^{-t\lambda} f(\lambda).$$

Q spectral measure $\Leftrightarrow$ $\varphi|_{\mathbb{R}_+}$ is a representation $\Leftrightarrow$ $\mathcal{E}_0 \cong \widehat{\mathcal{E}}$ $\Leftrightarrow$ $\widehat{U} = \varphi|_{\mathbb{R}_+}$.

Theorem (OS-dequantization by dilation)

For every strongly continuous symmetric contraction semigroup $(C_t)_{t\geq 0}$ on $\mathcal{H}$, dilation of $\varphi(t) = C_{|t|}$ leads to a reflection positive one-parameter group $(U, \mathcal{E})$ with $(\widehat{U}, \widehat{\mathcal{E}}) \cong (C, \mathcal{H})$ and $\mathcal{E}_0$ cycloc,

$$\mathcal{H} \cong (\mathcal{E}_+)^\theta \cong \widehat{\mathcal{E}} \quad \text{and} \quad PU_tP^* = C_{|t|} \quad \text{for} \quad t \in \mathbb{R}.$$

Problem: In general $\mathcal{E}$ is rather large.

The basic building blocks

Dilation picture: For $C_t = e^{-t\lambda}$, $\lambda > 0$, on $\mathcal{H} = \mathbb{C}$, dilation of $\varphi(t) = e^{-\lambda|t|}$ leads to the Cauchy distribution

$$\mathcal{E} = L^2(\mathbb{R}, \nu_\lambda), \quad d\nu_\lambda(x) = \frac{\lambda}{\pi}\frac{dx}{\lambda^2 + x^2}, \quad (U_t f)(x) = e^{-itx} f(x),$$

$$(\theta f)(x) = f(-x), \quad \mathcal{E}_+ := \overline{\mathrm{span}\{U_t 1 \colon t > 0\}}.$$

Hardy space picture:

$$T_\lambda \colon \mathcal{E} = L^2(\mathbb{R}, \nu_\lambda) \to L^2(\mathbb{R}), \quad (T_\lambda f)(x) := \frac{1}{\sqrt{\pi}} \frac{\sqrt{\lambda}}{\lambda + ix} f(x)$$

is unitary intertwining operator. $T_\lambda(\mathcal{E}_+)$ is the Hardy space of the lower half plane $\mathbb{R} - i\mathbb{R}_+$ and

$$(\theta f)(x) = \frac{\lambda - ix}{\lambda + ix} f(-x).$$

General case: Reduction to 1-dim. case by spectral measure of $A \geq 0$ for $C_t = e^{-tA}$: $\mathcal{E} = L^2(\mathbb{R}, Q; \mathcal{H})$ for operator density $Q(x) = \frac{1}{\pi} A(A^2 + x^2)^{-1}$.

Path space constructions

Exs: Important classes of contraction semigroups arise on $\mathcal{H} = L^2(G, dg)$, G a locally compact group, dg right Haar measure, by

$$C_t f = f * \mu_t \quad \text{where} \quad \mu_{t+s} = \mu_t * \mu_s$$

and $(\mu_t)_{t>0}$ is a convolution semigroup of probability measures. Its euclidean realization can be obtained on the path group $G^{\mathbb{R}}$

$$\mathcal{E} = L^2(G^{\mathbb{R}}, \widetilde{P}_\mu), \qquad (U_t F)(\gamma) := F(\gamma(\cdot + t)), \gamma \colon \mathbb{R} \to G$$
$$\mathcal{E}_0 = L^2(G) \cong \mathrm{ev}_0^* L^2(G) \subseteq L^2(G^{\mathbb{R}}, \widetilde{P}_\mu).$$

Here $\widetilde{P}_\mu = dg \otimes P_\mu$ according to $G^{\mathbb{R}} \cong G \times G^{\mathbb{R}}_*$, $G^{\mathbb{R}}_* = \{\gamma \colon \gamma(0) = e\}$ and P_μ is the unique probability measure on $G^{\mathbb{R}}_*$ corresponding to the two-sided Markov process with transition probabilites μ_t.

Ex: $G = \mathbb{R}^n$, $C_t = e^{t\Delta}$ the heat semigroup,
P_μ Wiener measure on $G^{\mathbb{R}}_*$.

Path models in QFT: Nelson ('64), A. Klein, L. Landau (1970s)

Definition (Symmetric Lie group)

A symmetric Lie group is a triple (G, H, τ), where τ is an involutive automorphism of G and $H \subseteq G^\tau$ an open subgroup.

Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ (τ-eigenspaces for ± 1)
Dual Lie algebra $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$
(G^c, H, τ) a dual Lie group if $\mathbf{L}(G^c) = \mathfrak{g}^c$

Example

(1) $\theta \colon \mathbb{R}^d \to \mathbb{R}^d, \theta(x) = (-x_0, x_1, \ldots, x_{d-1})$ (time reflection)
$\mathrm{O}_d(\mathbb{R})$ with $\tau(g) = \theta g \theta$ is dual to $L^\uparrow = \mathrm{O}_{1,d-1}(\mathbb{R})_+$ (Lorentz group)

(2) $\mathrm{Mot}(\mathbb{R}^d) = \mathbb{R}^d \rtimes \mathrm{O}_d(\mathbb{R})$ (euclidean motion group),
$\tau(x, g) = (\theta x, \theta g \theta)$
$\mathrm{Mot}(\mathbb{R}^d)^c = \mathbb{R}^d \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_+$ (Poincaré group)

(3) Heisenberg algebra: $\mathfrak{g} = \langle P, Q, Z \rangle$, $[P, Q] = Z$,
$\tau(Q) = Q, \tau(P) = -P, \tau(Z) = -Z \quad \Rightarrow \quad \mathfrak{g}^c \cong \mathfrak{g}$ (selfdual).

(4) $ax + b$-group: $G = \mathbb{R} \rtimes \mathbb{R}^\times$, $\tau(b, a) = (-b, a)$ with $G^c \cong \mathbb{R} \rtimes \mathbb{R}^\times_+$ (selfdual).

Definition (Reflection positive unitary representations)

(a) A reflection positive representation of the symmetric Lie group (G, H, τ) on $(\mathcal{E}, \mathcal{E}_+, \theta)$ is a unitary representation $(U, \mathcal{E})$ of G with

$$\theta U(g)\theta = U(\tau g) \quad \text{for} \quad g \in G \qquad \text{and} \qquad U(H)\mathcal{E}_+ = \mathcal{E}_+$$

(b) If $S \subseteq G$ is a subsemigroup invariant under $s^\sharp := \tau(s)^{-1}$, then U is reflection positive for (G, S, τ) if, in addition, $U(S)\mathcal{E}_+ \subseteq \mathcal{E}_+$.

Remark: (a) Then $\widehat{U}(h) := \widehat{U(h)}$ is a unitary representation of H on $\widehat{\mathcal{E}}$.
(b) If $x \in \mathfrak{q}$ is such that $U_t := U(\exp tx)$ preserves $\mathcal{E}_+$ for $t > 0$, then $\widehat{U}_t = e^{-tA_x}$ for $A_x = A_x^* \geq 0$ on $\widehat{\mathcal{E}}$.
(c) If $\mathcal{D} \subseteq \mathcal{E}_+ \cap \mathcal{E}^\infty$ (smooth vectors) is invariant under the derived representation $(\mathrm{d}U)(\mathfrak{g})$ and $\widehat{\mathcal{D}} = q(\mathcal{D}) \subseteq \widehat{\mathcal{E}}$ is dense, then

$$(\mathrm{d}U)^c(x + iy) := \widehat{\mathrm{d}U(x)} + i\widehat{\mathrm{d}U(y)}, \quad x \in \mathfrak{h}, y \in \mathfrak{q}$$

is an infinitesimally unitary representation of $\mathfrak{g}^c$.

Integrability problem: When is there a unitary representation $(U^c, \widehat{\mathcal{E}})$ of the 1-connected dual group G^c with $\mathrm{d}U^c = (\mathrm{d}U)^c$?

Reflection positive distributions

Let M be a manifold and $D \in \mathcal{D}'(M \times M)$ positive definite distribution.

$$\langle \varphi, \psi \rangle_D = \int_{M\times M} \varphi(x)\overline{\psi(y)} D(x,y)\,dx\,dy$$

defines by completion of $\mathcal{D}(M)$ a Hilbert space $\mathcal{E} := \mathcal{H}_D \subseteq \mathcal{D}'(M)$.

If $\theta \colon M \to M$ is an involution, $\theta^* D = D$ and $M_+ \subseteq M$ is open such that $D_+(x,y) := D(\theta x, y)$ is positive definite on M_+, then we call D reflection positive w.r.t. (M, M_+, θ). Then $\mathcal{E}_+ := \overline{\mathcal{D}(M_+)} \subseteq \mathcal{E}$ is θ-positive.

If the symmetric Lie group (G, H, τ) acts on M such that D is G-invariant, $\theta(g.x) = \tau(g).\theta(x)$ and $H.M_+ \subseteq M_+$, then the action of G on $\mathcal{D}(M)$ leads to a reflection positive unitary representation U_D on $(\mathcal{E}, \mathcal{E}_+, \theta)$ with $\widehat{\mathcal{E}} \cong \mathcal{H}_{D_+}$.

Ex: $M = \mathbb{R}^d, M_+ = \mathbb{R}^d_+$ (half space), $D(x,y) = \|x-y\|^{-a}$, $a \in [d-2, d)$, $G = \mathrm{Mot}(\mathbb{R}^d)$, $H = \mathrm{Mot}(\mathbb{R}^{d-1})$, $G^c =$ Poincaré group.

Theorem (General Integrability (Merigon, Ólafsson, N.; '14))

Let M_+ be a smooth manifold and D_+ a positive definite distribution on $M_+ \times M_+$. Let $\sigma\colon \mathfrak{g} \to \mathcal{V}(M_+)$ be such that $\sigma|_{\mathfrak{h}}$ integrates to an H-action and $\mathcal{L}^1_x D_+ = -\mathcal{L}^2_{\tau(x)} D_+$ for $x \in \mathfrak{g}$. Then there exists a unique unit. rep. $(\pi^c, \mathcal{H}_{D_+})$ of the simply connected Lie group G^c with $\mathtt{d}\pi^c(x) = \mathcal{L}_x$ (Lie derivative) for $x \in \mathfrak{g}^c$ on $\mathcal{H}_{D_+} \subseteq C^{-\infty}(M_+)$.

Key ingredient: J. Fröhlich's selfadjointness criterion (1980).
Ex.: $(G, H, \tau) = (\mathbb{R}^n, \{0\}, -\mathrm{id})$, $\sigma(v) = \frac{\partial}{\partial v}$
M_+ convex, $D_+(x,y) = \int_{\mathbb{R}^n} e^{-\lambda(x+y)}\, d\mu(\lambda)$, μ pos. meas.

Theorem (Refl.-Pos. Integrability (Merigon, Ólafsson, N.; '13))

If the reflection positive distribution D on (M, M_+, θ) and (G, H, τ) satisfy: D is G-invariant, $\theta(g.x) = \tau(g).\theta(x)$ and $H.M_+ \subseteq M_+$, then the reflection positive representation $(\pi, \mathcal{E} = \mathcal{H}_D, \mathcal{E}_+)$ of G leads to a unitary representation U^c of G^c on $\widehat{\mathcal{E}} \cong \mathcal{H}_{D_+}$ with $\mathtt{d}U^c(x + iy) = \widehat{\mathtt{d}U(x)} + i\widehat{\mathtt{d}U(y)}$ for $x \in \mathfrak{h}, y \in \mathfrak{q}$.

The $ax+b$-group

$G = \mathbb{R} \rtimes \mathbb{R}^\times$, $(b,a)(b',a') = (b+ab', aa')$
$\tau(b,a) = (-b,a)$, $S = \mathbb{R}_+ \rtimes \mathbb{R}_+^\times$ open subsemigroup
G has a unique irreducible representation (non-trivial on translations)

$$\mathcal{E} = L^2(\mathbb{R}), \quad (U(b,a)f)(x) = e^{-ibx}\sqrt{|a|}f(ax), \quad \theta = U(0,-1).$$

For $0 < s < 1$, the functions $|x|^{-s} \in \mathcal{E}^{-\infty}$ yield by smearing with $C_c^\infty(S)$ a θ-positive closed subspace $\mathcal{E}_+$, hence U is refl. pos. for (G,S,τ).
We thus obtain by OS-quantization for $G^c = \mathbb{R} \rtimes \mathbb{R}_+^\times$ (conn. component)

$$\widehat{\mathcal{E}} \cong L^2\Big(\mathbb{R}_+, \frac{dx}{x^{1-s}}\Big), \quad U^c(b,a) = e^{ibx}a^{s/2}f(ax).$$

Theorem

Every unitary representation of the connected Lie group $G^c \cong G_0$ can be obtained by OS-quantization from a reflection positive representation of G.

Reflection positivity for the Heisenberg group

Heisenberg algebra: $\mathfrak{g} = \langle P, Q, Z \rangle = \mathfrak{heis}(\mathbb{R}^2)$, $[P, Q] = Z$,
$\tau(Q) = Q, \tau(P) = -P, \tau(Z) = -Z \quad \Rightarrow \quad \mathfrak{g}^c \cong \mathfrak{g}$ (selfdual).

Schrödinger rep.: $(U_\lambda, L^2(\mathbb{R}))$, $\lambda \neq 0$, with

$$\mathtt{d}U_\lambda(Z) = i\lambda\mathbf{1}, \quad \mathtt{d}U_\lambda(Q) = ix, \quad \mathtt{d}U_\lambda(P) = \lambda\frac{d}{dx}.$$

The direct integral representation

$$\mathcal{E} := \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) \frac{d\lambda}{1+\lambda^2}, \quad (U(g)f)(\lambda) := U_\lambda(g)f(\lambda)$$

is reflection positive with

$$\theta(f)(\lambda) = f(-\lambda) \qquad \mathcal{E}_+ = \{f \in \mathcal{E} \colon \operatorname{supp}(\widehat{f}) \subseteq]-\infty, 0]\}.$$

It is obtained by dilation of the contr. semigroup $C_t f = e^{-t} f$ on $L^2(\mathbb{R})$.

Theorem

The corresponding dual representation $U^c \cong U_1$ on $\widehat{\mathcal{E}} \cong L^2(\mathbb{R})$ exists and coincides with the Schrödinger representation for $\lambda = 1$.

Scalar positive energy representations of Poincaré group

$\Omega = \{x \in \mathbb{R}^d \colon [x,x] > 0, x_0 > 0\}$ forward light cone
$H_m = \{x \in \mathbb{R}^d \colon [x,x] = m^2, x_0 > 0\}$ carries Lorentz inv. measure μ_m
General Lorentz invariant (tempered) measure $\mu = c\delta_0 + \int_0^\infty \mu_m \, d\rho(m)$
$\mathcal{H} := L^2(\overline{\Omega}, \mu)$ carries contraction semigroup $(C_t f)(x) = e^{-tx_0} f(x)$
and unitary rep. $(U^c_\mu(b,g)f)(x) = e^{-i[b,x]} f(g^{-1}x)$ of $G^c := \mathbb{R}^d \rtimes L^\uparrow$.

Theorem

(a) (Free scalar fields of mass m—one part. space) For $\mu = \mu_m$ the rep. U^c_μ is irreducible and the dilation space $\mathcal{E} \cong L^2(\mathbb{R}^d, \frac{dx}{m^2+x^2})$ of $(C, \mathcal{H})$ carries a reflection positive unitary rep. U of $G := \mathrm{Mot}(\mathbb{R}^d)$ w.r.t. time reflection $\theta(x_0, \mathbf{x}) = (-x_0, \mathbf{x})$ such that $\widehat{U} = U^c_\mu$.

(b) (Generalized free fields; $d \geq 3$—one part. space)
If $\delta(x) := \int_0^\infty \frac{1}{x^2+m^2} \, d\rho(m) < \infty$ for $x \neq 0$, then $\mathcal{E} := L^2(\mathbb{R}^d, \delta \cdot dx)$ carries a reflection pos. rep. U of G with $\widehat{\mathcal{E}} = L^2(\overline{\Omega}, \mu)$ and $\widehat{U} = U^c_\mu$.

Conformal invariance

Riesz measures on $\overline{\Omega}$: $\mu = \int_0^\infty \mu_m m^{1-s}\, dm$, $s < 2$,
$L^2(\overline{\Omega}, \mu)$ carries an irred. rep. U_s^c of the conformal group
$\mathrm{SO}_{2,d}(\mathbb{R})_0 = \mathrm{SO}_{1,d+1}(\mathbb{R})^c$ (highest weight rep.= pos. energy).

Theorem (Reflection positivity of complementary series)

(i) *For $0 < s \leq 2$, the distribution $D(x,y) = \frac{1}{\|x-y\|^{d-s}}$ on $\mathbb{R}^{2d}$ is reflection pos. for $(\mathbb{R}^d, \mathbb{R}^d_+, \theta)$.*

(ii) *The reflection pos. rep. of $\mathrm{Mot}(\mathbb{R}^d)$ on $\mathcal{H}_D \cong L^2(\mathbb{R}^d, \frac{dx}{\|x\|^s})$ extends to a compl. series rep U_s of the conformal group $\mathrm{SO}_{1,d+1}(\mathbb{R})$.*

(iii) *U_s is reflection positive w.r.t. the conformal compression semigroup S of $\mathbb{R}^d_+$ and $\widehat{U_s} = U_s^c$.*

Rem: This complements older results of Jorgensen/Ólafsson (1998) on reflection positivity for complementary series of Cayley type groups.

Conclusion

- OS-duality relates reflection positive representations of (G, S, τ) with representations of G^c where spectra of certain elements in $i\mathfrak{q}$ are positive.
  Typical: complementary series $\leftrightarrow$ holomorphic discrete series.
- The dilation construction is an amazingly effective tool to reverse the OS-quantization procedure. Is there a multidimensional analog?
- Are there Lie group versions of the path space models arising in QFT? (jt. with P. Jorgensen and G. Ólafsson).
- Are there interesting reflection positive representations of infinite-dimensional Lie groups? (such as the $\mathrm{Diff}(\mathbb{S}^1)$, loop groups)
- For inf. dim. motion groups this requires a better understanding of appropriate concepts of distributions on infinite-dimensional spaces.