Ladder operators for solvable potentials connected with exceptional orthogonal polynomials

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Exceptional orthogonal polynomials

- Joint work with Ian Marquette (U. of Queensland, Brisbane).

- **Exceptional orthogonal polynomials (EOP)**, introduced by Gómez-Ullate, Kamran, Milson in 2008, are complete and orthogonal polynomial systems generalizing the COP and such that
  - there are some gaps in the sequence of their degrees;
  - they occur in bound-state wf of some SUSYQM partners of shape invariant potentials, which are rational extensions of the latter.

- In \( n \)th-order SUSYQM, one starts from \( n \) different seed solutions \( \phi_1, \phi_2, \ldots, \phi_n \) of a starting Hamiltonian \( H^{(1)} \) and one gets the potential of the partner \( H^{(2)} \) as \( V^{(2)}(x) = V^{(1)}(x) - 2 \frac{d^2}{dx^2} W(\phi_1, \phi_2, \ldots, \phi_n) \), provided the Wronskian is nonsingular.

- In general, 3 types of EOP (I, II, III). For I or II, \( H^{(2)} \) is strictly isospectral to \( H^{(1)} \) and shape invariant. For III, there are additional levels below the spectrum of \( H^{(1)} \) and \( H^{(2)} \) is not shape invariant (the only case for HO).
Ladder operators of $H^{(1)}$

- Assume that $H^{(1)}$ has ladder operators $a$ and $a^{\dagger}$ of order $k$. These generate with $H^{(1)}$ a polynomial Heisenberg algebra (PHA) of order $k - 1$

$$[H^{(1)}, a] = -\lambda a, \quad [H^{(1)}, a^{\dagger}] = \lambda a^{\dagger},$$

$$[a, a^{\dagger}] = P^{(1)}(H^{(1)} + \lambda) - P^{(1)}(H^{(1)}),$$

with $\lambda = \text{constant}$ and $P^{(1)}(H^{(1)}) = k$th-degree polynomial in $H^{(1)}$

$$P^{(1)}(H^{(1)}) = \prod_{i=1}^{k}(H^{(1)} - \epsilon_i).$$

- $a$ and/or $a^{\dagger}$ may have zero modes $\Rightarrow$ this PHA may have $\infty$-dim unirreps, as well as finite-dim ones (singlet, doublet, . . . , multiplet).

- Examples
  - HO : $a = \frac{d}{dx} + x$, $k = 1$, $\lambda = 2$, $P^{(1)}(H^{(1)}) = H^{(1)} - 1 \Rightarrow$ Heisenberg algebra ;
  - RHO : $a = \frac{1}{4} \left(2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} + \frac{1}{2}x^2 - \frac{2\ell(\ell+1)}{x^2} + 1\right)$, $k = 2$, $\lambda = 2$
    $$P^{(1)}(H^{(1)}) = \frac{1}{16}(2H^{(1)} - 3 - 2\ell)(2H^{(1)} - 1 + 2\ell) \Rightarrow su(1,1) \text{ Lie algebra.}$$
Ladder operators of $H^{(2)}$ in SUSYQM

- In $n$th-order SUSYQM, $H^{(1)}$ and $H^{(2)}$ intertwine with $n$th-order differential operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ (written in terms of the $\varphi_i$’s). These supercharges enable to relate the energy spectra, the wfs and the ladder operators of $H^{(1)}$ and $H^{(2)}$.

- Ladder operators of $H^{(2)}$, $b = \mathcal{A}a\mathcal{A}^\dagger$, $b^\dagger = \mathcal{A}a^\dagger\mathcal{A}^\dagger$ are $(2n + k)$th-order differential operators. With $H^{(2)}$, they generate another PHA,

\[
[H^{(2)}, b] = -\lambda b, \quad [H^{(2)}, b^\dagger] = \lambda b^\dagger, \\
[b, b^\dagger] = P^{(2)}(H^{(2)} + \lambda) - P^{(2)}(H^{(2)}),
\]

with $P^{(2)}(H^{(2)}) = (2n + k)$th-degree polynomial in $H^{(2)}$, such that

\[
P^{(2)}(H^{(2)}) = P^{(1)}(H^{(2)}) f(H^{(2)} - \lambda) f(H^{(2)}), \\
f(H^{(2)}) = \mathcal{A}\mathcal{A}^\dagger.
\]

4
Ladder operators for EOP-related problems

- In JMP 54 (2013) 042102, we used this method to construct ladder operators for potentials related to (type III) Hermite and (type I, II, or III) Laguerre EOP.

- For type I or II, corresponding to isospectrality, the properties of \((H^{(1)}, a, a^\dagger)\) are transferred to \((H^{(2)}, b, b^\dagger)\) and \(b, b^\dagger\) act on all the wfs of \(H^{(2)}\).

- For type III, although the properties of \((H^{(1)}, a, a^\dagger)\) are still transferred to \((H^{(2)}, b, b^\dagger)\), the operators \(b, b^\dagger\) do not see the states that have been added below the spectrum of \(H^{(1)}\). Hence, apart from some unirreps similar to those of \(H^{(1)}\), there are also \(n\) singlets corresponding to the added states.

- For reasons that will appear in Ian’s talk on superintegrability, it is desirable to get other ladder operators than \(b, b^\dagger\) that will mix all states together.

- In a first attempt for \(n = 2\) (JPA 46 (2013) 155201), we proposed other ladder operators for which the two added states form a doublet.
Type III : state adding versus state deleting


- On slide 2, $H^{(2)}$ was obtained by adding some states below the spectrum of $H^{(1)}$: state adding (or Darboux-Crum) approach. Here $\varphi_1, \varphi_2, \ldots, \varphi_n$ are unphysical solutions of the S.E. for $H^{(1)}$ with energy less than the ground-state one, which are converted into physical solutions of $H^{(2)}$.

- Another possibility for getting $H^{(2)}$: state deleting (or Krein-Adler) approach. Here $\varphi_1, \varphi_2, \ldots, \varphi_n$ are $n$ excited bound-state wfs of $H^{(1)}$, whose energy is removed from the spectrum (note that in general the two $n$ values will be different).

- To distinguish both possibilities:
  1) $(H^{(1)}, H^{(2)})$ with $H^{(i)} = -\frac{d^2}{dx^2} + V^{(i)}(x), \quad i = 1, 2;
  2) (\bar{H}^{(1)}, \bar{H}^{(2)})$ with $\bar{H}^{(i)} = -\frac{d^2}{dx^2} + \bar{V}^{(i)}(x), \quad i = 1, 2.$

- For the HO and the RHO cases, it is possible to arrive at $\bar{V}^{(2)} = V^{(2)} + \text{constant}$, thereby allowing the construction of ladder operators.
Harmonic oscillator case

- We start from the same HO Hamiltonian: \( \bar{H}^{(1)} = H^{(1)} \) (or \( \bar{V}^{(1)} = V^{(1)} = x^2, -\infty < x < \infty \)).

- In the state adding case, take \( n \to k \) and

\[
\varphi_i(x) = \phi_{m_i}(x) = \mathcal{H}_{m_i}(x)e^{x^2/2}, \quad \mathcal{H}_{m_i}(x) = (-i)^{m_i} H_{m_i}(ix),
\]

\( m_1 < m_2 < \cdots < m_k \) \( \) with \( m_i \) even (resp. odd) for \( i \) odd (resp. even);

then \( \bar{E}_\nu^{(2)} = 2\nu + 1, \nu = -m_k - 1, \ldots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \ldots. \)

- In the state deleting case, take \( n \to m_k - k + 1 \) and

\[
(\varphi_1, \varphi_2, \ldots, \varphi_n) \rightarrow (\psi_1, \psi_2, \ldots, \tilde{\psi}_{m_k-m_k-1}, \ldots, \tilde{\psi}_{m_k-m_2}, \ldots, \tilde{\psi}_{m_k-m_1}, \ldots, \psi_{m_k});
\]

then \( \bar{E}_\nu^{(2)} = 2m_k + 2\nu + 3, \nu = -m_k - 1, \ldots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \ldots. \)

- Result: \( \bar{V}^{(2)} = V^{(2)} + 2m_k + 2. \)

- Example: \( n = k = 1, m_1 = 2 \): add state corresponding to \( \nu = -3 \) or delete excited states with \( \nu = 1 \) and 2 \( \Rightarrow \) same spectrum apart from translation (see following slide).
Energy spectrum of $H^{(2)}$ and action of $c^\dagger$ on the eigenstates for $k = 1, m_1 = 2$. The $\nu$ values are indicated on the right.
Ladder operators for harmonic oscillator

• New operators $c = \bar{A}A^\dagger$ and $c^\dagger = A\bar{A}^\dagger$ of order $m_k + 1$:

\[
\begin{align*}
H^{(2)} &\xrightarrow{A^\dagger} H^{(1)} = \bar{H}^{(1)} \xrightarrow{\bar{A}} H^{(2)} = H^{(2)} + 2m_k + 2 \\
\Rightarrow c &\quad \text{lowering operator for } H^{(2)}.
\end{align*}
\]

• Corresponding PHA of $m_k$th order:

\[
\begin{align*}
[H^{(2)}, c] &= -(2m_k + 2)c, & [H^{(2)}, c^\dagger] &= (2m_k + 2)c^\dagger, \\
[c, c^\dagger] &= Q(H^{(2)} + 2m_k + 2) - Q(H^{(2)}), & Q(H^{(2)}) &= c^\dagger c,
\end{align*}
\]

\[
Q(H^{(2)}) = \left( \prod_{i=1}^k (H^{(2)} + 2m_i + 1) \right) \left( \prod_{j=1}^{m_k} \prod_{j\neq m_k-m_{k-1},...,m_k-m_1} (H^{(2)} - 2j - 1) \right).
\]

• Action of $Q(H^{(2)})$ on $\psi^{(2)}_\nu(x)$ obtained by replacing $H^{(2)}$ by $E^{(2)}_\nu \Rightarrow$ action of $c$ and $c^\dagger$ on $\psi^{(2)}_\nu(x) \Rightarrow$ PHA has $m_k + 1 \infty$-dim unirreps (see example on slide8).
Radial harmonic oscillator case

• More complicated because SUSYQM changes the $\ell$ value in $V_\ell(x) = \frac{1}{4}x^2 + \frac{\ell(\ell+1)}{x^2} (0 < x < \infty)$.

• In the state adding case, take $n \to k$ and $V^{(1)}(x) = V_{\ell+k}(x)$, so that $V^{(2)}(x) = \text{rationally-extended } V_\ell(x)$ and $E^{(2)}_{\ell,\nu} = 2\nu + \ell + k + \frac{3}{2}$, $\nu = -m_k - 1, \ldots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \ldots$.

• In the state deleting case, take $n \to m_k + 1 - k$ and $\bar{V}^{(1)}(x) = V_{\ell+k-m_k-1}(x)$, where $\ell + k$ is assumed greater than $m_k + 1$. Then $\bar{V}^{(2)}(x) = \text{rationally-extended } V_\ell(x) = V^{(2)}(x) + m_k + 1$ and $\bar{E}^{(2)}_{\ell,\nu} = 2\nu + \ell + k + m_k + \frac{5}{2}$, $\nu = -m_k - 1, \ldots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \ldots$.

• Here $\bar{H}^{(1)} \neq H^{(1)} \Rightarrow$ need of a third SUSYQM transformation of supercharges $\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^\dagger$ relating them: $\bar{H}^{(1)} = \tilde{H}^{(1)} \to \tilde{H}^{(2)} = H^{(1)} + m_k + 1$, by using $m_k + 1$ seed solutions of class II.

• Example: $k = 2, m_1 = 2, m_2 = 3$ : add states corresponding to $\nu = -3$ and $\nu = -4$ or delete excited states with $\nu = 2$ and $\nu = 3 \Rightarrow$ same spectrum up to a translation (see following figure).

10
Fig. 2. Energy spectrum of $H^{(2)}$ and action of $c^\dagger$ on the eigenstates for $m_1 = 2$, $m_2 = 3$, $\ell = 2$. The $\nu$ values are indicated on the right.
Ladder operators for radial harmonic oscillator

- New operators \( c = \tilde{\mathcal{A}}\mathcal{A}\tilde{\mathcal{A}}\dagger, \ c\dagger = \mathcal{A}\tilde{\mathcal{A}}\dagger\tilde{\mathcal{A}}\dagger \) of order \( 2m_k + 2 \).
- Corresponding PHA of order \( 2m_k + 1 \) : same expression as for HO, except that

\[
Q(H^{(2)}) = \left( \prod_i (H^{(2)} - \ell + 2m_i - k + \frac{1}{2}) \right) \left( \prod_j (H^{(2)} + \ell - 2j + k - \frac{1}{2}) \right) \left( \prod_n (H^{(2)} - \ell - 2n - k - \frac{3}{2}) \right),
\]

where now \( i = 1, 2, \ldots, k \), \( j = 0, 1, \ldots, m_k \), and \( n = 1, 2, \ldots, m_k \) with the exceptions of \( n = m_k - m_{k-1}, \ldots, m_k - m_2, m_k - m_1 \).

- Resulting action of \( c \) and \( c\dagger \) is not substantially different from that for the HO ⇒ PHA has \( m_k + 1 \) \( \infty \)-dim unirreps (see example on slide 11).
Conclusion

• We have shown that for the multi-step extensions of the HO and RHO connected with type III EOP, it is possible to build new ladder operators $c$, $c^\dagger$ that satisfy PHA with only $\infty$-dim unirreps. These new ladder operators differ from the usual ones $b$, $b^\dagger$, well known in SUSYQM.

• This is made possible by combining the state adding (or Darboux-Crum) and state-deleting (or Krein-Adler) approaches to the construction of these extensions.

• As it will be shown by Ian Marquette, such new ladder operators are very useful to construct integrals of motion for superintegrable systems based on one-dimensional multi-step extensions connected with type III EOP.

• Interesting open question: is such a construction possible for other types of potentials than the HO and the RHO, such as potentials connected with Jacobi type III EOP?

THANK YOU!