Ladder operators for solvable potentials connected with exceptional orthogonal polynomials

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Exceptional orthogonal polynomials

- Joint work with Ian Marquette (U. of Queensland, Brisbane).
- Exceptional orthogonal polynomials (EOP), introduced by Gómez-Ullate, Kamran, Milson in 2008, are complete and orthogonal polynomial systems generalizing the COP and such that
  - there are some gaps in the sequence of their degrees ;
  - they occur in bound-state wf of some SUSYQM partners of shape invariant potentials, which are rational extensions of the latter.
- In nth-order SUSYQM, one starts from n different seed solutions φ_1, φ_2, $\ldots$, φ_n of a starting Hamiltonian $H^{(1)}$ and one gets the potential of the partner $H^{(2)}$ as $V^{(2)}(x) = V^{(1)}(x) - 2\frac{d^2}{dx^2}\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$, provided the Wronskian is nonsingular.
- In general, 3 types of EOP (I, II, III). For I or II, $H^{(2)}$ is strictly isospectral to $H^{(1)}$ and shape invariant. For III, there are additional levels below the spectrum of $H^{(1)}$ and $H^{(2)}$ is not shape invariant (the only case for HO).

Ladder operators of $H^{(1)}$

- Assume that $H^{(1)}$ has ladder operators a and $a^\dagger$ of order k. These generate with $H^{(1)}$ a polynomial Heisenberg algebra (PHA) of order $k-1$

$$[H^{(1)}, a] = -\lambda a, \qquad [H^{(1)}, a^\dagger] = \lambda a^\dagger,$$
$$[a, a^\dagger] = P^{(1)}(H^{(1)} + \lambda) - P^{(1)}(H^{(1)}),$$

  with $\lambda =$ constant and $P^{(1)}(H^{(1)}) =$ kth-degree polynomial in $H^{(1)}$ $= \prod_{i=1}^{k}(H^{(1)} - \epsilon_i)$.

- a and/or $a^\dagger$ may have zero modes $\Rightarrow$ this PHA may have ∞-dim unirreps, as well as finite-dim ones (singlet, doublet, …, multiplet).

- Examples
  - HO : $a = \frac{d}{dx} + x$, $k = 1$, $\lambda = 2$, $P^{(1)}(H^{(1)}) = H^{(1)} - 1 \Rightarrow$ Heisenberg algebra ;
  - RHO : $a = \frac{1}{4}\left(2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \frac{1}{2}x^2 - \frac{2\ell(\ell+1)}{x^2} + 1\right)$, $k = 2$, $\lambda = 2$, $P^{(1)}(H^{(1)}) = \frac{1}{16}(2H^{(1)} - 3 - 2\ell)(2H^{(1)} - 1 + 2\ell) \Rightarrow$ su(1,1) Lie algebra.

Ladder operators of $H^{(2)}$ in SUSYQM

- In nth-order SUSYQM, $H^{(1)}$ and $H^{(2)}$ intertwine with nth-order differential operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ (written in terms of the φ_i's). These supercharges enable to relate the energy spectra, the wfs and the ladder operators of $H^{(1)}$ and $H^{(2)}$.
- Ladder operators of $H^{(2)}$, $b = \mathcal{A} a \mathcal{A}^\dagger$, $b^\dagger = \mathcal{A} a^\dagger \mathcal{A}^\dagger$ are $(2n+k)$th-order differential operators. With $H^{(2)}$, they generate another PHA,

$$[H^{(2)}, b] = -\lambda b, \qquad [H^{(2)}, b^\dagger] = \lambda b^\dagger,$$
$$[b, b^\dagger] = P^{(2)}(H^{(2)} + \lambda) - P^{(2)}(H^{(2)}),$$

with $P^{(2)}(H^{(2)}) =$ $(2n+k)$th-degree polynomial in $H^{(2)}$, such that

$$P^{(2)}(H^{(2)}) = P^{(1)}(H^{(2)}) f(H^{(2)} - \lambda) f(H^{(2)}),$$
$$f(H^{(2)}) = \mathcal{A}\mathcal{A}^\dagger.$$

Ladder operators for EOP-related problems

- In JMP 54 (2013) 042102, we used this method to construct ladder operators for potentials related to (type III) Hermite and (type I, II, or III) Laguerre EOP.
- For type I or II, corresponding to isospectrality, the properties of $(H^{(1)}, a, a^{\dagger})$ are transferred to $(H^{(2)}, b, b^{\dagger})$ and b, $b^{\dagger}$ act on all the wfs of $H^{(2)}$.
- For type III, although the properties of $(H^{(1)}, a, a^{\dagger})$ are still transferred to $(H^{(2)}, b, b^{\dagger})$, the operators b, $b^{\dagger}$ do not see the states that have been added below the spectrum of $H^{(1)}$. Hence, apart from some unirreps similar to those of $H^{(1)}$, there are also n singlets corresponding to the added states.
- For reasons that will appear in Ian's talk on superintegrability, it is desirable to get other ladder operators than b, $b^{\dagger}$ that will mix all states together.
- In a first attempt for $n = 2$ (JPA 46 (2013) 155201), we proposed other ladder operators for which the two added states form a doublet.

Type III : state adding versus state deleting

- New attempt : construct new ladder operators c, $c^\dagger$ (JMP 54 (2013) 102102 and arXiv :1402.6380, submitted to JMP).
- On slide 2, $H^{(2)}$ was obtained by adding some states below the spectrum of $H^{(1)}$: state adding (or Darboux-Crum) approach. Here φ_1, φ_2, ..., φ_n are unphysical solutions of the S.E. for $H^{(1)}$ with energy less than the ground-state one, which are converted into physical solutions of $H^{(2)}$.
- Another possibility for getting $H^{(2)}$: state deleting (or Krein-Adler) approach. Here φ_1, φ_2, ..., φ_n are n excited bound-state wfs of $H^{(1)}$, whose energy is removed from the spectrum (note that in general the two n values will be different).
- To distinguish both possibilities :

  1) $(H^{(1)}, H^{(2)})$ with $H^{(i)} = -\frac{d^2}{dx^2} + V^{(i)}(x), \quad i = 1, 2\,;$

  2) $(\bar{H}^{(1)}, \bar{H}^{(2)})$ with $\bar{H}^{(i)} = -\frac{d^2}{dx^2} + \bar{V}^{(i)}(x), \quad i = 1, 2.$
- For the HO and the RHO cases, it is possible to arrive at $\bar{V}^{(2)} = V^{(2)} +$ constant, thereby allowing the construction of ladder operators.

Harmonic oscillator case

- We start from the same HO Hamiltonian : $\bar{H}^{(1)} = H^{(1)}$ (or $\bar{V}^{(1)} = V^{(1)} = x^2$, $-\infty < x < \infty$).
- In the state adding case, take $n \to k$ and

$$\varphi_i(x) = \phi_{m_i}(x) = \mathcal{H}_{m_i}(x) e^{x^2/2}, \qquad \mathcal{H}_{m_i}(x) = (-\mathrm{i})^{m_i} H_{m_i}(\mathrm{i}x),$$
$$m_1 < m_2 < \cdots < m_k \quad \text{with } m_i \text{ even (resp. odd) for } i \text{ odd (resp. even);}$$

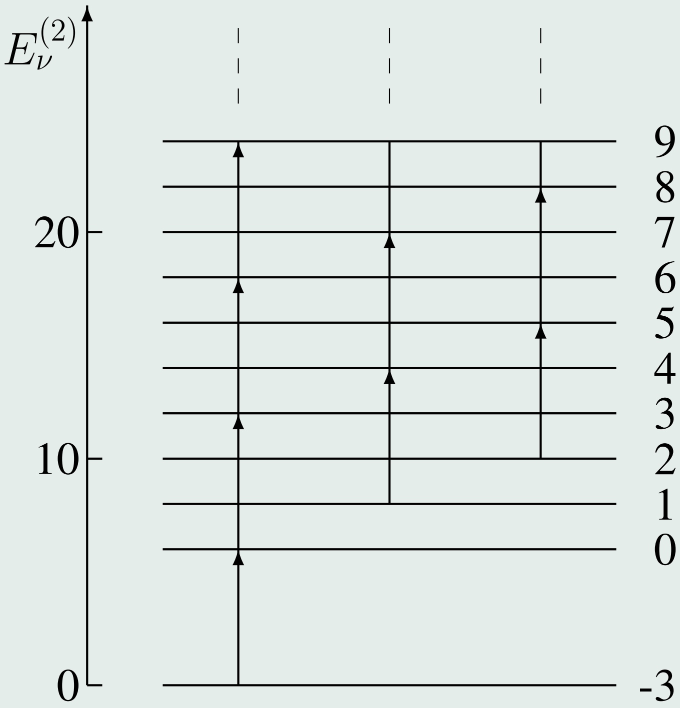
then $E^{(2)}_\nu = 2\nu + 1$, $\nu = -m_k - 1, \ldots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \ldots$.

- In the state deleting case, take $n \to m_k - k + 1$ and

$$(\varphi_1, \varphi_2, \ldots, \varphi_n) \to$$
$$(\psi_1, \psi_2, \ldots, \check{\psi}_{m_k - m_{k-1}}, \ldots, \check{\psi}_{m_k - m_2}, \ldots, \check{\psi}_{m_k - m_1}, \ldots, \psi_{m_k});$$

then $\bar{E}^{(2)}_\nu = 2m_k + 2\nu + 3$, $\nu = -m_k - 1, \ldots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \ldots$.

- Result : $\bar{V}^{(2)} = V^{(2)} + 2m_k + 2$.
- Example : $n = k = 1$, $m_1 = 2$: add state corresponding to $\nu = -3$ or delete excited states with $\nu = 1$ and $2 \Rightarrow$ same spectrum apart from translation (see following slide).



Energy spectrum of $H^{(2)}$ and action of $c^\dagger$ on the eigenstates for $k = 1, m_1 = 2$. The ν values are indicated on the right.

Ladder operators for harmonic oscillator

- New operators $c = \bar{\mathcal{A}}\mathcal{A}^\dagger$ and $c^\dagger = \mathcal{A}\bar{\mathcal{A}}^\dagger$ of order $m_k + 1$:

$$H^{(2)} \xrightarrow{\mathcal{A}^\dagger} H^{(1)} = \bar{H}^{(1)} \xrightarrow{\bar{\mathcal{A}}} \bar{H}^{(2)} = H^{(2)} + 2m_k + 2$$

$$H^{(2)} \xrightarrow{c} H^{(2)} + 2m_k + 2$$

$\Rightarrow c =$ lowering operator for $H^{(2)}$.

- Corresponding PHA of m_kth order :

$$[H^{(2)}, c] = -(2m_k + 2)c, \qquad [H^{(2)}, c^\dagger] = (2m_k + 2)c^\dagger,$$
$$[c, c^\dagger] = Q(H^{(2)} + 2m_k + 2) - Q(H^{(2)}), \qquad Q(H^{(2)}) = c^\dagger c,$$

$$Q(H^{(2)}) = \left(\prod_{i=1}^{k}(H^{(2)} + 2m_i + 1)\right)\left(\prod_{\substack{j=1 \\ j \neq m_k - m_{k-1}, \ldots, m_k - m_1}}^{m_k}(H^{(2)} - 2j - 1)\right).$$

- Action of $Q(H^{(2)})$ on $\psi_\nu^{(2)}(x)$ obtained by replacing $H^{(2)}$ by $E_\nu^{(2)} \Rightarrow$ action of c and $c^\dagger$ on $\psi_\nu^{(2)}(x) \Rightarrow$ PHA has $m_k + 1$ ∞-dim unirreps (see example on slide8).

Radial harmonic oscillator case

- More complicated because SUSYQM changes the ℓ value in $V_\ell(x) = \frac{1}{4}x^2 + \frac{\ell(\ell+1)}{x^2}$ $(0 < x < \infty)$.
- In the state adding case, take $n \to k$ and $V^{(1)}(x) = V_{\ell+k}(x)$, so that $V^{(2)}(x) =$ rationally-extended $V_\ell(x)$ and $E^{(2)}_{\ell,\nu} = 2\nu + \ell + k + \frac{3}{2}$, $\nu = -m_k - 1, \ldots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \ldots$.
- In the state deleting case, take $n \to m_k + 1 - k$ and $\bar{V}^{(1)}(x) = V_{\ell+k-m_k-1}(x)$, where $\ell+k$ is assumed greater than m_k+1. Then $\bar{V}^{(2)}(x) =$ rationally-extended $V_\ell(x) = V^{(2)}(x) + m_k + 1$ and $\bar{E}^{(2)}_{\ell,\nu} = 2\nu + \ell + k + m_k + \frac{5}{2}$, $\nu = -m_k - 1, \ldots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \ldots$.
- Here $\bar{H}^{(1)} \neq H^{(1)} \Rightarrow$ need of a third SUSYQM transformation of supercharges $\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^\dagger$ relating them : $\bar{H}^{(1)} = \tilde{H}^{(1)} \to \tilde{H}^{(2)} = H^{(1)} + m_k + 1$, by using $m_k + 1$ seed solutions of class II.
- Example : $k = 2$, $m_1 = 2$, $m_2 = 3$: add states corresponding to $\nu = -3$ and $\nu = -4$ or delete excited states with $\nu = 2$ and $\nu = 3 \Rightarrow$ same spectrum up to a translation (see following figure).

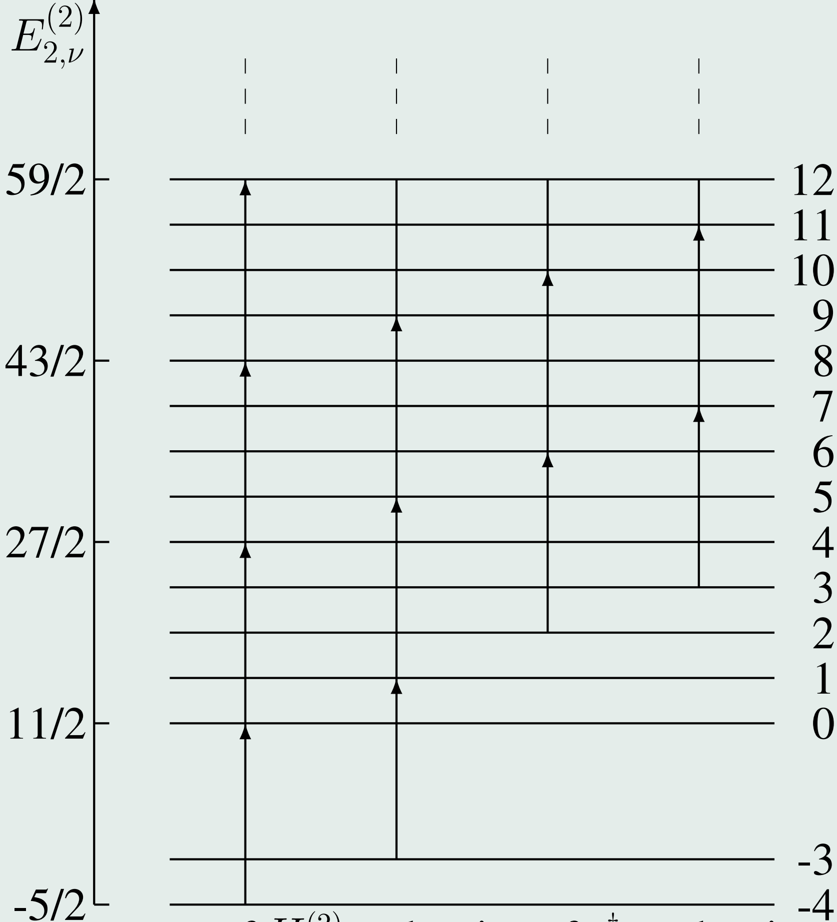


Fig. 2. Energy spectrum of $H^{(2)}$ and action of $c^\dagger$ on the eigenstates for $m_1 = 2$, $m_2 = 3$, $\ell = 2$. The ν values are indicated on the right.

Ladder operators for radial harmonic oscillator

- New operators $c = \bar{\mathcal{A}}\tilde{\mathcal{A}}\mathcal{A}^\dagger$, $c^\dagger = \mathcal{A}\tilde{\mathcal{A}}^\dagger\bar{\mathcal{A}}^\dagger$ of order $2m_k + 2$.
- Corresponding PHA of order $2m_k + 1$: same expression as for HO, except that

$$Q(H^{(2)}) = \left(\prod_i (H^{(2)} - \ell + 2m_i - k + \tfrac{1}{2})\right) \left(\prod_j (H^{(2)} + \ell - 2j + k - \tfrac{1}{2})\right) \left(\prod_n (H^{(2)} - \ell - 2n - k - \tfrac{3}{2})\right),$$

where now $i = 1, 2, \ldots, k$, $j = 0, 1, \ldots, m_k$, and $n = 1, 2, \ldots, m_k$ with the exceptions of $n = m_k - m_{k-1}, \ldots, m_k - m_2, m_k - m_1$.

- Resulting action of c and $c^\dagger$ is not substantially different from that for the HO $\Rightarrow$ PHA has $m_k + 1$ ∞-dim unirreps (see example on slide 11).

Conclusion

- We have shown that for the multi-step extensions of the HO and RHO connected with type III EOP, it is possible to build new ladder operators c, $c^\dagger$ that satisfy PHA with only ∞-dim unirreps. These new ladder operators differ from the usual ones b, $b^\dagger$, well known in SUSYQM.
- This is made possible by combining the state adding (or Darboux-Crum) and state-deleting (or Krein-Adler) approaches to the construction of these extensions.
- As it will be shown by Ian Marquette, such new ladder operators are very useful to construct integrals of motion for superintegrable systems based on one-dimensional multi-step extensions connected with type III EOP.
- Interesting open question : is such a construction possible for other types of potentials than the HO and the RHO, such as potentials connected with Jacobi type III EOP ?

THANK YOU !