Group foliation of finite difference equations

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Group foliation

Group foliation:

Method for obtaining invariant, partially invariant, and non-invariant solutions of G-invariant differential equations.

Idea:

Solutions to $\Delta = 0$ are obtained by solving the resolving and automorphic systems.

Resolving system:

"Projection" of $\Delta = 0$ onto the space of invariants.

Automorphic system:

All solutions lie on a single orbit of G.

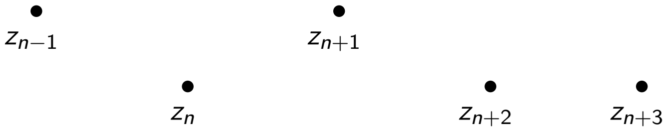
Historical overview

Group foliation:

- 1895: Lie laid out the basic ideas in 2 examples
- 1904: Vessiot formalized Lie's ideas
- 1969 –: Equations from fluid dynamics (Ovsiannikov and Soviet mathematicians)
- 2001: Heavenly and complex Monge–Ampère equations (Martina, Nutku, Sheftel, and Winternitz)
- 2005/08: EDS formulation (Anderson, Fels, and Pohjanpelto)
- 20??: Moving frame formulation (Thompson – V)
- Today: Finite difference equations

Geometric setting

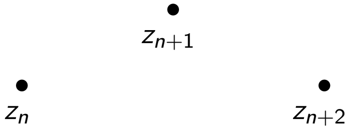
Consider $\{z_n = (x_n, y_n) \,|\, n \in \mathbb{Z}\}$:



k^{th} order forward discrete jets:

$$J^{[k]} = \{z_n^{[k]} = (n, z_n, z_{n+1}, \ldots, z_{n+k})\}$$

$z_n^{[2]}$:



Finite difference equations

A (forward) finite difference equation is

$$E(z_n^{[k]}) = E(n, z_n, \ldots, z_{n+k}) = 0.$$

In many applications

$$E = 0 \quad \longrightarrow \quad \Delta = 0$$

in the continuous limit.

Running example

$$\left(\frac{x_{n+1}^{a+1}}{a+1}+\frac{y_{n+1}^{1-b}}{b-1}\right)-\left(\frac{x_n^{a+1}}{a+1}+\frac{y_n^{1-b}}{b-1}\right)+k(x_{n+1}-x_n)=0$$

$$x_{n+1}-x_n=h$$

$a\neq -1,\ b\neq 1.$

Approximates

$$y_x=(k+x^a)y^b$$

Symmetry

Assume

$$E(z_n^{[k]}) = 0$$

admits a symmetry group.

A Lie group G is a symmetry group of $E(z_n^{[k]}) = 0$ if and only if

$$E(g \cdot z_n^{[k]}) = 0 \qquad \text{whenever} \qquad E(z_n^{[k]}) = 0,$$

where

$$Z_n^{[k]} = g \cdot z_n^{[k]} = (n, g \cdot z_n, \ldots, g \cdot z_{n+k}).$$

Example

The numerical scheme

$$\left(\frac{y_{n+1}^{1-b}}{b-1}+\frac{x_{n+1}^{a+1}}{a+1}\right)-\left(\frac{y_n^{1-b}}{b-1}+\frac{x_n^{a+1}}{a+1}\right)+k(x_{n+1}-x_n)=0 \qquad x_{n+1}-x_n=h$$

is invariant under $G=(\mathbb{R},+)$:

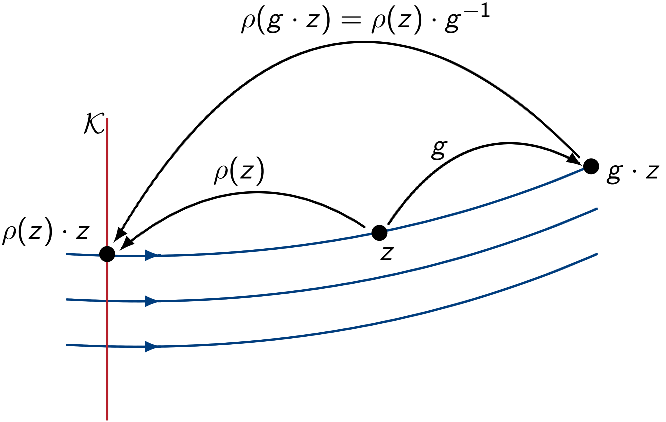
$$X_n = x_n + \epsilon \qquad \frac{Y_n^{1-b}}{b-1} = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1} - \frac{(x_n+\epsilon)^{a+1}}{a+1}$$

$$X_{n+1} = x_{n+1} + \epsilon \qquad \frac{Y_{n+1}^{1-b}}{b-1} = \frac{y_{n+1}^{1-b}}{b-1} + \frac{x_{n+1}^{a+1}}{a+1} - \frac{(x_{n+1}+\epsilon)^{a+1}}{a+1}$$

Joint moving frames

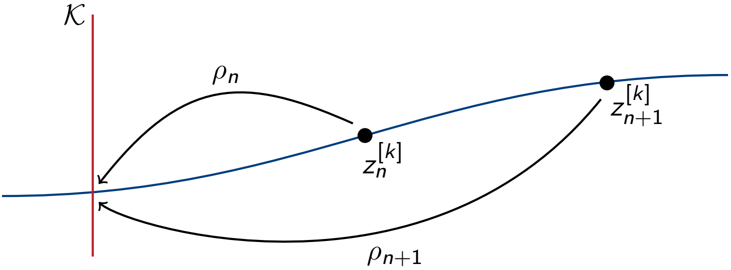
A (right) joint moving frame is a map $\rho\colon J^{[k]} \to G$ satisfying

$$\rho(g \cdot z_n^{[k]}) = \rho(z_n^{[k]}) \cdot g^{-1}$$



$z = z_n^{[k]}$

Joint moving frames



$$\rho_n = \rho(z_n^{[k]}) \qquad \rho_{n+1} = \rho(z_{n+1}^{[k]})$$

Construction

- Product action

$$X_n = x_n + \epsilon, \qquad \frac{Y_n^{1-b}}{b-1} = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1} - \frac{(x_n+\epsilon)^{a+1}}{a+1}$$

- Choose a cross-section

$$\mathcal{K} = \{x_n = 0\}$$

- Solve the normalizing equations

$$0 = X_n = x_n + \epsilon \qquad \Rightarrow \qquad \rho_n : \quad \epsilon_n = -x_n$$

Invariantization

The invariantization of z_m w.r.t. $\rho_n = \rho(z_n^{[k]})$ is the invariant

$$\iota_n(z_m) = \rho_n \cdot z_m$$

Proof: $\iota_n(g \cdot z_m) = \rho(g \cdot z_n^{[k]}) \cdot g \cdot z_m = \rho(z_n^{[k]}) \cdot g^{-1} \cdot g \cdot z_m = \iota_n(z_m)$

$$H_n = \iota_n(x_{n+1}) = x_{n+1} + \epsilon_n \Big|_{\epsilon_n = -x_n} = x_{n+1} - x_n$$

$$J_n = \iota_n\left(\frac{y_n^{1-b}}{b-1}\right) = \frac{y_n^{1-b}}{b-1} + \frac{x_n^{a+1}}{a+1}$$

$$K_n = \iota_n\left(\frac{y_{n+1}^{1-b}}{b-1}\right) = \frac{y_{n+1}^{1-b}}{b-1} + \frac{x_{n+1}^{a+1}}{a+1} - \frac{(x_{n+1} - x_n)^{a+1}}{a+1}$$

Question

$$\iota_n(z_m) = \rho_n \cdot z_m$$

$$\iota_{n+1}(z_m) = \rho_{n+1} \cdot z_m$$

are invariants. For example

$$\iota_n(x_{n+1}) = H_n = x_{n+1} - x_n$$

$$\iota_{n+1}(x_{n+1}) = 0$$

How are they "related"?

Recurrence relation

The recurrence relation between $\iota_n(z_m)$ and $\iota_{n+1}(z_m)$ is

$$\iota_n(z_m) = \mathcal{Q}_n \cdot \iota_{n+1}(z_m),$$

where $\mathcal{Q}_n = \rho_n \cdot \rho_{n+1}^{-1}$ [=(Joint) Maurer–Cartan invariant].

Proof:

$$\iota_n(z_m) = \rho_n \cdot z_m = \rho_n \cdot \rho_{n+1}^{-1} \cdot \rho_{n+1} \cdot z_m = \mathcal{Q}_n \cdot \iota_{n+1}(z_m)$$

In general,

$$\iota_n(z_m) = \mathcal{Q}_n \cdot \mathcal{Q}_{n+1} \cdots \mathcal{Q}_{n+k-1} \cdot \iota_{n+k}(z_m)$$

Example

For

$$\rho_n = \epsilon_n = -x_n$$

we have

$$\mathcal{Q}_n = \rho_n \cdot \rho_{n+1}^{-1} = \epsilon_n - \epsilon_{n+1} = -x_n + x_{n+1} = H_n$$

Symbolically!

$$H_n = \iota_n(x_{n+1}) = \mathcal{Q}_n \cdot \iota_{n+1}(x_{n+1}) = \mathcal{Q}_n \cdot 0 = \mathcal{Q}_n + 0 = \mathcal{Q}_n$$

Example

For

$$H_n = \iota_n(x_{n+1}) \qquad\qquad J_n = \iota_n(y_n)$$

we have

$$\iota_n(x_{n+2}) = \mathcal{Q}_n \cdot \iota_{n+1}(x_{n+2}) = \mathcal{Q}_n \cdot H_{n+1} = H_n \cdot H_{n+1} = H_n + H_{n+1}$$

$$\iota_n\left(\frac{y_{n+1}^{1-b}}{b-1}\right) = \mathcal{Q}_n \cdot \iota_{n+1}\left(\frac{y_{n+1}^{1-b}}{b-1}\right) = \mathcal{Q}_n \cdot J_{n+1} = H_n \cdot J_{n+1} = J_{n+1} - \frac{H_n^{a+1}}{a+1}$$

In general,

$$\iota_n(x_{n+k}) = \sum_{i=1}^{k} H_{n+i-1} \qquad \iota_n\left(\frac{y_{n+k}^{1-b}}{b-1}\right) = J_{n+k} - \frac{1}{a+1}\left(\sum_{i=1}^{k} H_{n+i-1}\right)^{a+1}$$

Algebra of joint invariants

A set of joint invariants $\mathbf{I}_{\text{gen}}$ is said to be a generating set if any joint invariant can be expressed in terms of $\mathbf{I}_{\text{gen}}$ and their shifts.

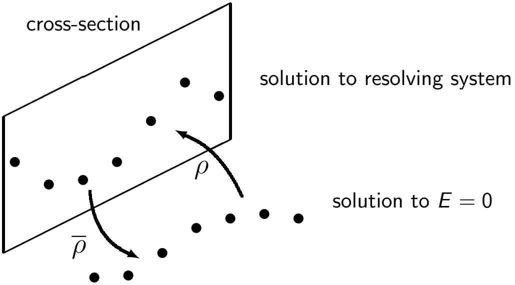
The algebra of joint invariants is generated by

$$\mathcal{Q}_n \qquad \text{and} \qquad \iota_n(z_n).$$

Group foliation

Given $E(z_n^{[k]}) = 0$ with symmetry group G

- solve the resolving system
  - projection of solution space onto the space of joint invariants
- solve the reconstruction equations
  - sends solutions of the resolving system to solutions of the original equation



Resolving system

- Find a minimal generating set $\mathbf{I}_{\text{gen}}$
- Express $E(z_n^{[k]})$ in terms of $\mathbf{I}_{\text{gen}}$ and its shifts

$$0 = E(z_n^{[k]}) = \widetilde{E}(\ldots \mathbf{I}_{\text{gen},\ell} \ldots)$$

Example

In terms of the generating invariants

$$H_n \qquad J_n$$

the equations

$$\left(\frac{y_{n+1}^{1-b}}{b-1}+\frac{x_{n+1}^{a+1}}{a+1}\right)-\left(\frac{y_n^{1-b}}{b-1}+\frac{x_n^{a+1}}{a+1}\right)+k(x_{n+1}-x_n)=0 \qquad x_{n+1}-x_n=h$$

are

$$J_{n+1}-J_n+k\,H_n=0 \qquad H_n=h$$

Their solution is

$$J_n=J_0-(k\,h)n \qquad H_n=h$$

Reconstruction equations

Solution in the space of invariants $\rightsquigarrow$ original solution

The reconstruction equations are

$$\overline{\rho}_{n+1} = \overline{\rho}_n \cdot \mathcal{Q}_n$$

The solution z_n to $E(z_n^{[k]}) = 0$ is

$$z_n = \overline{\rho}_n \cdot \iota_n(z_n)$$

Example

Let $\overline{\rho}_n = \overline{\epsilon}_n$. Since $\mathcal{Q}_n = H_n = h$,

$$\overline{\rho}_{n+1} = \overline{\rho}_n \cdot \mathcal{Q}_n \qquad \Rightarrow \qquad \overline{\epsilon}_{n+1} = \overline{\epsilon}_n + H_n = \overline{\epsilon}_n + h$$

so that $\overline{\epsilon}_n = h\, n + \overline{\epsilon}_0$. Since

$$\iota_n(x_n) = 0 \qquad\qquad \iota_n(y_n) = [(b-1)J_n]^{1/(1-b)}$$

we have

$$\begin{aligned} x_n &= \overline{\rho}_n \cdot 0 = h\, n + \overline{\epsilon}_0 \\ y_n &= \overline{\rho}_n \cdot [(b-1)J_n]^{1/(1-b)} = (1-b)^{1/(1-b)} \left[k\, x_n + \frac{x_n^{1+a}}{1+a} + C \right]^{1/(1-b)} \end{aligned}$$

where $C = -J_0 - k\,\epsilon_0$.

Ending remarks

- Symbolic computations: only requires G and $\mathcal{K}$.
- Method extends to multi-index finite difference equations:

$$\frac{u_{m+1,n+1} - u_{m+1,n} - u_{m,n+1} + u_{m,n}}{h\,k} = u_{m,n}$$

- Method produces new numerical schemes.