Gauge-covariant extensions

of Killing tensors

and conservation laws

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Noether's theorem

Classical and quantum-mechanical hamiltonian systems:

symmetries ⇔ conservation laws

Let $G(x,p)$ generate infinitesimal phase-space transformations

$$\delta x = \{x, G\} = \frac{\partial G}{\partial p}, \quad \delta p = \{p, G\} = -\frac{\partial G}{\partial x}, \text{ s.t. } \delta G = \{G, G\} = 0.$$

Then

$$\delta H = \{H, G\} = -\frac{dG}{dt} = 0$$

implies invariance of the hamiltonian action *mod* boundary terms.

$$\text{N.B.} \quad \delta S = \int_1^2 dt \left[\frac{d}{dt} \left(p \frac{\partial G}{\partial p} - G \right) + \{G, H\} \right]$$

Geodesic motion

Geodesic hamiltonian

$$H = \frac{1}{2} g^{\mu\nu}(x)\, p_\mu p_\nu \quad \Rightarrow \quad p_\mu = g_{\mu\nu}\, \dot{x}^\nu$$

constants of motion: $J(x,p) = J^\mu(x)\, p_\mu$

$$\{J, H\} = 0 \quad \Leftrightarrow \quad \nabla_\mu J_\nu + \nabla_\nu J_\mu = 0$$

i.e., J^μ is a Killing vector.

These constants of motion generate *isometries* on the *configuration space*:

$$\delta x^\mu = \frac{\partial J}{\partial p_\mu} = J^\mu(x), \qquad J^\lambda \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial J^\lambda}{\partial x^\mu} g_{\lambda\nu} + \frac{\partial J^\lambda}{\partial x^\nu} g_{\mu\lambda} = 0.$$

However

$$\delta p_\mu = -\frac{\partial J}{\partial x^\mu} = A_\mu^{\ \nu}(x)\, p_\nu, \quad \text{with} \quad A_\mu^{\ \nu} = -\frac{\partial J^\nu}{\partial x^\mu};$$

this transformation rule is *not* covariant ($A_\mu^{\ \nu}$) is not a tensor).

This can be mended by defining *covariant* infinitesimal variations

$$\Delta x^\mu = \delta x^\mu = \frac{\partial G}{\partial p_\mu}, \qquad \Delta p_\mu = \delta p_\mu - \delta x^\lambda\, \Gamma_{\lambda\mu}^{\ \ \nu}\, p_\nu \equiv -\mathcal{D}_\mu G$$

with a covariant derivative $\mathcal{D}$ such that for a generating function

$$\Delta G = \Delta x^\mu\, \mathcal{D}_\mu G + \Delta p_\mu \frac{\partial G}{\partial p_\mu} = \{G, G\} = 0$$

$$\Rightarrow \quad \mathcal{D}_\mu G = \frac{\partial G}{\partial x^\mu} + \Gamma_{\mu\nu}^{\ \ \lambda}\, p_\lambda \frac{\partial G}{\partial p_\nu}$$

In particular if $G = G^{\mu_1 \cdots \mu_n}(x)\, p_{\mu_1} ... p_{\mu_n},$

then $\mathcal{D}_\mu G = (\nabla_\mu G^{\mu_1 \cdots \mu_n})\, p_{\mu_1} ... p_{\mu_n}.$

Hence by metric postulate:

$$\mathcal{D}_\mu H = \frac{1}{2} \left(\nabla_\mu \, g^{\nu\lambda} \right) p_\nu p_\lambda = 0.$$

Now

$$\Delta H = \{H, G\} = \mathcal{D}_\mu H \, \frac{\partial G}{\partial p_\mu} - \frac{\partial H}{\partial p_\mu} \mathcal{D}_\mu G = -p^\mu \mathcal{D}_\mu G$$

$$\Delta H = 0 \quad \Leftrightarrow \quad \frac{1}{(n+1)!} \left(\nabla_{\mu_{n+1}} G_{\mu_1 \ldots \mu_n} \right) p^{\mu_1} ... p^{\mu_{n+1}} = 0.$$

The coefficient functions are *Killing tensors*.

Example: Kerr geometry

geodesic hamiltonian

$$H = \frac{1}{2\rho^2}\left[\Delta^2 p_r^2 + p_\theta^2 + \left(a \sin\theta\, p_t + \frac{p_\varphi}{\sin\theta}\right)^2 - \frac{1}{\Delta^2}\left((r^2+a^2)p_t + ap_\varphi\right)^2\right]$$

where

$$\Delta^2 = r^2 - 2Mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta.$$

Killing tensor = Carter's constant

$$K = \frac{1}{2\rho^2}\left[-\Delta^2 a^2 \cos^2\theta\, p_r^2 + r^2 p_\theta^2 + r^2 \sin^2\theta \left(ap_t + \frac{p_\varphi}{\sin^2\theta}\right)^2 + \frac{a^2\cos^2\theta}{\Delta^2}\left((r^2+a^2)p_t + ap_\varphi\right)^2\right].$$

Abelian gauge fields

Point mass m and charge q

$$H = \frac{1}{2m} g^{\mu\nu} (p_\mu - qA_\mu)(p_\nu - qA_\nu)$$

Equations of motion

$$p_\mu = m g_{\mu\nu} \dot{x}^\nu + qA_\mu, \qquad g_{\mu\nu} \left(\ddot{x}^\nu + \Gamma_{\kappa\lambda}^{\;\;\nu} \dot{x}^\kappa \dot{x}^\lambda \right) = \frac{q}{m} F_{\mu\nu} \dot{x}^\nu.$$

$\rightarrow$ Gauge transformations $A'_\mu = A_\mu + \nabla_\mu \Lambda$
also change canonical momentum: $p'_\mu = p_\mu + q\nabla_\mu \Lambda$.

Use covariant momentum $\pi_\mu = p_\mu - qA_\mu$:

$$H = \frac{1}{2m} g^{\mu\nu} \pi_\mu \pi_\nu$$

New covariant brackets:

$$\{G,K\} = \mathcal{D}_\mu G\,\frac{\partial K}{\partial \pi_\mu} - \frac{\partial G}{\partial \pi_\mu}\,\mathcal{D}_\mu K + qF_{\mu\nu}\,\frac{\partial G}{\partial \pi_\mu}\,\frac{\partial K}{\partial \pi_\nu}$$

with

$$\mathcal{D}_\mu G = \frac{\partial G}{\partial x^\mu} + \Gamma_{\mu\nu}{}^{\lambda}\,\pi_\lambda\,\frac{\partial G}{\partial \pi_\nu}$$

s.t.

$$\{x^\mu, \pi_\nu\} = \delta^\mu_\nu, \qquad \{\pi_\mu, \pi_\nu\} = qF_{\mu\nu},$$

and

$$\dot{G} = \{G, H\}\,.$$

As before

$$G(x,\pi) = G^{\mu_1 \cdots \mu_n}(x)\,\pi_{\mu_1} \ldots \pi_{\mu_n} \quad \Rightarrow \quad \mathcal{D}_\mu G = (\nabla_\mu G^{\mu_1 \cdots \mu_n})\,\pi_{\mu_1} \ldots \pi_{\mu_n}$$

Constants of motion

Conservation law: $\{G, H\} = 0$

$$\Rightarrow \quad \pi^{\mu} \mathcal{D}_{\mu} G = q \pi^{\mu} F_{\mu\nu} \frac{\partial G}{\partial \pi_{\nu}}$$

Non-homogeneous expressions in momentum:

$$G(x, \pi) \;=\; \sum_{n} \frac{1}{n!} G^{(n)\,\mu_1 \ldots \mu_n}(x)\, \pi_{\mu_1} \ldots \pi_{\mu_n}$$

In components:

$$\nabla_{(\mu_1} G^{(n)}_{\mu_2 \ldots \mu_{n+1})} = q F_{(\mu_1}{}^{\nu} G^{(n+1)}_{\mu_2 \ldots \mu_{n+1})\nu}$$

Coefficient functions are generalized Killing tensors

If the metric admits a Killing tensor $G^{(n)}_{\mu_1\ldots\mu_n}$ then take all

$$G^{(k)}_{\mu_1\ldots\mu_k} = 0, \quad \forall k \geq n+1,$$

and compute the lower components by the generalized Killing equations:

$$\nabla_\mu G^{(0)} = qF_\mu{}^\nu G^{(1)}_\nu,$$

$$\nabla_\mu G^{(1)}_\nu + \nabla_\nu G^{(1)}_\mu = qF_\mu{}^\lambda G^{(2)}_{\lambda\nu} + qF_\nu{}^\lambda G^{(2)}_{\lambda\mu},$$

$$\nabla_\mu G^{(2)}_{\nu\lambda} + \nabla_\nu G^{(2)}_{\lambda\mu} + \nabla_\lambda G^{(2)}_{\mu\nu} = qF_\mu{}^\kappa G^{(3)}_{\kappa\nu\lambda} + qF_\nu{}^\kappa G^{(3)}_{\kappa\lambda\mu} + qF_\lambda{}^\kappa G^{(3)}_{\kappa\mu\nu},$$

$$\ldots$$

plus terms involving a scalar potential, when applicable

Example: quantum-dot model

2-particle system in magnetic field $F_{ij} = \varepsilon_{ijk} B_k$ and Coulomb and harmonic potential Φ separable in CM co-ordinates:

$$H_{CM} = \frac{1}{2} g^{ij} \pi_i \pi_j + \Phi(\rho, z, \varphi),$$

with

$$g_{ij} = \text{diag}(1, 1, \rho^2), \qquad \Phi = \frac{1}{2}\left(\omega_0^2 \rho^2 + \omega_z^2 z^2\right) - \frac{\kappa}{\sqrt{\rho^2 + z^2}}.$$

and brackets

$$\{G, K\} = \mathcal{D}_i G \frac{\partial K}{\partial \pi_i} - \frac{\partial G}{\partial \pi_i} \mathcal{D}_i K - 2\omega_L \rho \left(\frac{\partial G}{\partial \pi_\rho} \frac{\partial K}{\partial \pi_\varphi} - \frac{\partial G}{\partial \pi_\varphi} \frac{\partial K}{\partial \pi_\rho} \right)$$

where $\omega_L = eB/2$ is the Larmor frequency.

System possesses a rank-4 Killing tensor, from which a constant of motion can be constructed if the magnetic field is tuned to give $\omega_L^2 + \omega_0^2 = 4\omega_z^2$:

$$
\begin{aligned}
G \;=\; & \rho^2\pi_z^4 - 2\rho z\pi_\rho\pi_z^3 + z^2\pi_\rho^2\pi_z^2 + \frac{1}{\rho^2}\pi_\varphi^4 + \pi_\rho^2\pi_\varphi^2 + \left(2 + \frac{z^2}{\rho^2}\right)\pi_z^2\pi_\varphi^2 \\
& + 2\omega_L\pi_\varphi\left(\rho^2\pi_\rho^2 + (2\rho^2 + z^2)\pi_z^2\right) + \left[(2\omega_z^2 - \omega_0^2)z^2\rho^2 + 2\omega_L^2\rho^4 - \frac{2\kappa\rho^2}{\sqrt{\rho^2+z^2}}\right]\pi_z^2 \\
& + \left[2\omega_z^2 z^3\rho + \frac{2\kappa z\rho}{\sqrt{\rho^2+z^2}}\right]\pi_\rho\pi_z + \omega_L^2\rho^4\pi_\rho^2 + \left[2\omega_z^2 z^2 + (\omega_0^2 - 5\omega_L^2)\rho^2 - \frac{2\kappa}{\sqrt{\rho^2+z^2}}\right]\pi_\varphi^2 \\
& - 2\omega_L\pi_\varphi\left[(3\omega_L^2 - \omega_0^2)\rho^4 - 2\omega_z^2\rho^2 z^2 + \frac{2\kappa\rho^2}{\sqrt{\rho^2+z^2}}\right] + \omega_z^4\rho^2 z^4 + 2\omega_z^2\omega_L^2\rho^4 z^2 \\
& - \omega_L^2(3\omega_L^2 - 4\omega_z^2)\rho^6 + \frac{2\kappa}{\sqrt{\rho^2+z^2}}\left(\omega_z^2\rho^2 z^2 - \omega_L^2\rho^4\right) + \frac{\kappa^2}{2}\frac{\rho^2 - z^2}{\rho^2+z^2}.
\end{aligned}
$$

Non-abelian gauge fields

Non-abelian Lorentz force

$$g_{\mu\nu}\left(\ddot{x}^{\nu}+\Gamma_{\kappa\lambda}^{\;\;\nu}\dot{x}^{\kappa}\dot{x}^{\lambda}\right)=\frac{g}{m}\,t_a F^a_{\mu\nu}\,\dot{x}^{\nu}, \qquad \dot{t}_a+g f_{ab}^{\;\;\;c}\,t_c\,A^b_{\mu}\,\dot{x}^{\mu}=0.$$

where

$$F^a_{\mu\nu}=\nabla_{\mu}A^a_{\nu}-\nabla_{\nu}A^a_{\mu}+g f_{bc}^{\;\;\;a}A^b_{\mu}A^c_{\nu}.$$

Canonical hamiltonian:

$$H=\frac{1}{2}\,g^{\mu\nu}\pi_{\mu}\pi_{\nu},$$

Brackets:

$$\{G,K\} = \mathcal{D}_\mu G \frac{\partial K}{\partial \pi_\mu} - \frac{\partial G}{\partial \pi_\mu}\mathcal{D}_\mu K + g t_a F^a_{\mu\nu} \frac{\partial G}{\partial \pi_\mu}\frac{\partial K}{\partial \pi_\nu} + f_{ab}{}^c t_c \frac{\partial G}{\partial t_a}\frac{\partial K}{\partial t_b}.$$

$$\mathcal{D}_\mu G \equiv \frac{\partial G}{\partial x^\mu} + \Gamma_{\mu\nu}{}^\lambda \pi_\lambda \frac{\partial G}{\partial \pi_\nu} + g f_{ab}{}^c\, t_c\, A^a_\mu \frac{\partial G}{\partial t_b}$$

Ricci identity:

$$\{\pi_\mu, \pi_\nu\} = g t_a F^a_{\mu\nu},$$

Algebra of gauge charges:

$$\{t_a, t_b\} = f_{ab}{}^c t_c.$$

Constants of motion

$$\{G, H\} = 0 \quad \Rightarrow \quad \pi^{\mu} \mathcal{D}_{\mu} G = g t_a F^{a}_{\mu\nu} \frac{\partial G}{\partial \pi_{\nu}}$$

In components:

$$\mathcal{D}_{\mu} G^{(0)} = g t_a F_{\mu}^{a\,\nu} G^{(1)}_{\nu},$$

$$\mathcal{D}_{\mu} G^{(1)}_{\nu} + \mathcal{D}_{\nu} G^{(1)}_{\mu} = g t_a \left(F_{\mu}^{a\,\lambda} G^{(2)}_{\lambda\nu} + F_{\nu}^{a\,\lambda} G^{(2)}_{\lambda\mu} \right),$$

$$\mathcal{D}_{\mu} G^{(2)}_{\nu\lambda} + \mathcal{D}_{\nu} G^{(2)}_{\lambda\mu} + \mathcal{D}_{\lambda} G^{(2)}_{\mu\nu} = g t_a \left(F_{\mu}^{a\,\kappa} G^{(3)}_{\kappa\lambda\nu} + F_{\nu}^{a\,\kappa} G^{(2)}_{\kappa\lambda\mu} + F_{\lambda}^{a\,\kappa} G^{(3)}_{\kappa\mu\nu} \right),$$

$$\ldots$$

Example: 2-D SU(2) Yang-Mills theory

Non-abelian magnetic field $F_{ij}^a = \varepsilon_{ij} B^a$ satisfying YM-equation

$$\nabla_i B^a + g\,(A_i \times B)^a = 0 \quad \Rightarrow \quad \nabla_i |B|^2 = 0,$$

and by gauge rotation can take $B^a = constant$. Vector potential:

$$A_i^a = -\frac{1}{2}\,\varepsilon_{ij}\,x^j B^a.$$

In euclidean plane there is a conserved Runge-Lenz vector:

$$\begin{aligned} K_i &= x_i \boldsymbol{\pi}^2 - \pi_i \boldsymbol{\pi} \cdot \mathbf{x} + g B^a t_a \left(\frac{1}{2}\,\varepsilon_{ij} \pi_j \mathbf{x}^2 + x_i x_j \varepsilon_{jk} \pi_k \right) \\ &\quad + \frac{1}{2}\,(g B^a t_a)^2\, x_i\, \mathbf{x}^2. \end{aligned}$$

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