

On Mathieu Moonshine

GROUP 30: 30TH ICGTMP, Ghent, Belgium, July 14-18, 2014

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Plan:

- 1 Mathieu Moonshine: The problem
- 2 The solution: Facts and Fantasies
- 3 The evidence: Massless representation of M_{24}

- [Taormina/W11] *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}* , JHEP **1308**:152 (2013); arXiv:1107.3834 [hep-th]
- [Taormina/W12] *A twist in the M_{24} moonshine story*; arXiv:1303.3221 [hep-th]
- [Taormina/W13] *Symmetry-surfing the moduli space of Kummer $K3$ s*; arXiv:1303.2931 [hep-th]
- [W14] *Snapshots of conformal field theory*;
to appear in "Mathematical Aspects of Quantum Field Theories",
Mathematical Physics Studies, Springer; arXiv:1404.3108 [hep-th]

1. Mathieu Moonshine on K3: The observation

M : a K3 surface, that is, a cpct. Calabi-Yau 2-fold, $h^{1,0}(M) = 0$

ELLIPTIC GENUS of M for $\tau, z \in \mathbb{C}$, $\text{Im}(\tau) > 0$:

$$\mathcal{E}_{K3}(\tau, z) = 8 \left(\frac{\vartheta_2(\tau, z)}{\vartheta_2(\tau, 0)} \right)^2 + 8 \left(\frac{\vartheta_3(\tau, z)}{\vartheta_3(\tau, 0)} \right)^2 + 8 \left(\frac{\vartheta_4(\tau, z)}{\vartheta_4(\tau, 0)} \right)^2.$$

For each $N = (2, 2)$ SCFT at central charges $c = \bar{c} = 6$ with space-time SUSY and integral $U(1)$ charges:

Its CFT elliptic genus

$$\mathcal{E}_{CFT}(\tau, z) = s \text{Tr}_{\mathcal{H}_R} (y^{J_0} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}), \quad (q = e^{e\pi i \tau}, y = e^{2\pi i z})$$

\mathcal{H}_R : Ramond sector,

J_0, L_0, \bar{L}_0 : zero modes of the $U(1)$ -current and Virasoro fields in the SCA

either vanishes, or it agrees with $\mathcal{E}_{K3}(\tau, z)$.

The theory has $N = (4, 4)$ SUSY.

Examples: orbifolds of toroidal SCFTs, Gepner models at $c = \bar{c} = 6$

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Definition (K3 THEORY)

An $N = (4, 4)$ SCFT at $c = \bar{c} = 6$ with space-time SUSY, integral $U(1)$ charges and CFT elliptic genus $\mathcal{E}_{K3}(\tau, z)$.

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1. Decomposition into irreducible $N = 4$ characters

3 types of $N = 4$ irreps \mathcal{H}_\bullet with $\chi_\bullet(\tau, z) = \text{sTr}_{\mathcal{H}_\bullet}(y^{J_0} q^{L_0 - 1/4})$:

- vacuum \mathcal{H}_0 with $\chi_0(\tau, 0) = -2$
- massless matter $\mathcal{H}_{1/2}$ with $\chi_{1/2}(\tau, 0) = 1$
- massive matter \mathcal{H}_h ($h \in \mathbb{R}_{>0}$), $\chi_h(\tau, z) = q^h \tilde{\chi}(\tau, z)$, $\chi_h(\tau, 0) = 0$

Ansatz: $\mathcal{H}_R = \mathcal{H}_0 \otimes \bar{\mathcal{H}}_0 \oplus 20 \mathcal{H}_{1/2} \otimes \bar{\mathcal{H}}_{1/2}$
 $\oplus (\bigoplus_{0 < n \in \mathbb{N}} [f_n \mathcal{H}_n \otimes \bar{\mathcal{H}}_0 \oplus \bar{f}_n \mathcal{H}_0 \otimes \bar{\mathcal{H}}_n])$
 $\oplus (\bigoplus_{0 < m \in \mathbb{N}} [g_m \mathcal{H}_m \otimes \bar{\mathcal{H}}_{1/2} \oplus \bar{g}_m \mathcal{H}_{1/2} \otimes \bar{\mathcal{H}}_m])$
 $\oplus \bigoplus_{0 < h, \bar{h} \in \mathbb{R}} k_{h, \bar{h}} \mathcal{H}_h \otimes \bar{\mathcal{H}}_{\bar{h}}$

where all $f_n, \bar{f}_n, g_m, \bar{g}_m, k_{h, \bar{h}}$ are non-negative integers.

$$\Rightarrow \mathcal{E}_{K3}(\tau, z) = -2\chi_0(\tau, z) + 20\chi_{1/2}(\tau, z) + 2e(\tau)\tilde{\chi}(\tau, z),$$

$$2e(\tau) = \sum_{n=1}^{\infty} (g_n - 2f_n)q^n$$

Conjecture [Eguchi/Ooguri/Tachikawa10] For all n , $g_n - 2f_n$ gives the dimension of a non-trivial representation of the Mathieu group M_{24} .

2. Solving Mathieu Moonshine? – Facts

Theorem [Gannon12] using results of **Cheng, Duncan, Gaberdiel, Hohenegger, Persson, Ronellenfitsch, Volpato**

There **exists** a **representation** \mathcal{R}_n of M_{24} for every $n \in \mathbb{N}$, s.th.

$$\mathcal{H}_{R,gen}^{BRST} \cong (-2)\mathcal{H}_0 \oplus 20 \mathcal{H}_{1/2} \oplus \bigoplus_{n=1}^{\infty} \mathcal{R}_n \otimes \mathcal{H}_n$$

as a **representation** of M_{24} and of the $N=4$ superconformal algebra.

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WHY?

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WHY?

HOW?

Is there an underlying structure of a **vertex algebra**?

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Theorem [Mukai88]

If G is a symmetry group of a K3 surface M ,

that is, G fixes the two-forms that define the hyperkähler structure of M ,

then G is isomorphic to a subgroup of the Mathieu group M_{24} ,
and $|G| \leq 960 \ll 244.823.040 = |M_{24}|$.

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Theorem [Mukai88]

If G is a **symmetry group** of a K3 surface M ,

that is, G **fixes** the **two-forms** that define the **hyperkähler structure** of M ,

then G is isomorphic to a **subgroup** of the Mathieu group M_{24} ,
and $|G| \leq 960 \ll 244.823.040 = |M_{24}|$.

[Gaberdiel/Hohenegger/Volpato11]

M_{24} **cannot** act as **symmetry group** of a **K3 theory**.

2. Solving Mathieu Moonshine? – Facts and Fantasies

Observation [Taormina/W10-13]

The map $\mathcal{H}_R \rightarrow \mathcal{H}_{R,gen}^{BRST}$ depends on the choice of a geometric interpretation; SO: restrict to geometric symmetry groups.

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The map $\mathcal{H}_R \rightarrow \mathcal{H}_{R,gen}^{BRST}$ depends on the choice of a geometric interpretation; SO: restrict to geometric symmetry groups.

Conjecture [Taormina/W10-13]

In every geometric interpretation,

$$\mathcal{H}_{R,gen}^{BRST} \cong (-2)\mathcal{H}_0 \oplus \mathcal{R}_{1/2} \otimes \mathcal{H}_{1/2} \oplus \bigoplus_{n=1}^{\infty} \mathcal{R}_n \otimes \mathcal{H}_n$$

as a representation of the geometric symmetry group $G \subset M_{24}$; the rhs collects the symmetries from distinct points of the moduli space.

We call this procedure **SYMMETRY SURFING**.

3. Symmetries of \mathbb{Z}_2 -orbifold CFTs on K3

G : **geometric** symmetry group of a \mathbb{Z}_2 -orbifold CFT
 with **geometric interpretation** on $X = T/\mathbb{Z}_2$, $T = \mathbb{C}^2/\Lambda$

using [Fujiki88]
 \implies

$$G = (\mathbb{Z}_2)^4 \rtimes G_T \subset (\mathbb{Z}_2)^4 \rtimes \text{GL}_4(\mathbb{F}_2) \stackrel{[\text{Jordan1870}]}{\cong} (\mathbb{Z}_2)^4 \rtimes A_8,$$

$$\mathbb{F}_2^4 \cong \frac{1}{2}\Lambda/\Lambda, \quad G_T \subset \text{SO}(3)$$

- $G_T \subset (G_T)_k$, one of three **maximal finite groups**
 $(G_T)_1 = A_4$, $(G_T)_2 = S_3$, $(G_T)_0 = \mathbb{Z}_2^2$
- our choice of **lattice generators** $\Lambda = \text{span}_{\mathbb{C}}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$
 for the corresponding **maximally symmetric** lattices Λ_k is

$$\Lambda_1 = \text{span}_{\mathbb{Z}}\{(1,0,0,0), (0,1,0,0), (0,0,1,0), \frac{1}{2}(1,1,1,1)\},$$

$$\Lambda_2 = \text{span}_{\mathbb{Z}}\{(1,0,0,0), \frac{1}{2}(-1, \sqrt{3}, 0, 0), (0,0,1,0), \frac{1}{2}(0,0, -1, \sqrt{3})\},$$

$$\Lambda_0 = \text{span}_{\mathbb{Z}}\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}.$$

3. Symmetry surfing the moduli space of Kummer K3s

Result [Taormina/W11&12]

For the \mathbb{Z}_2 -orbifold CFTs on K3 with geometric interpretation on some $X = \widetilde{T}/\mathbb{Z}_2$ with $T = \mathbb{C}^2/\Lambda$, the joint action of all symmetry groups yields the maximal subgroup $\text{Aff}(\mathbb{F}_2^4) = (\mathbb{Z}_2)^4 \rtimes A_8 \subset M_{24}$.

Note: $\mathbb{Z}_2^4 \rtimes A_8$ is not a subgroup of M_{23} .

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Note: $\mathbb{Z}_2^4 \rtimes A_8$ is not a subgroup of M_{23} .

Recall:

$$\mathcal{H}_{R,gen}^{BRST} \cong (-2)\mathcal{H}_0 \oplus \mathcal{R}_{1/2} \otimes \mathcal{H}_{1/2} \oplus \bigoplus_{n=1}^{\infty} \mathcal{R}_n \otimes \mathcal{H}_n$$

Result [Taormina/W13]

\mathcal{R}_1 can be constructed as a 90-dim. space of states common to all K3-theories that are \mathbb{Z}_2 -orbifolds of toroidal theories.

As common representation space of all geometric symmetry groups of Kummer K3s, \mathcal{R}_1 carries an action of $\mathbb{Z}_2^4 \rtimes A_8$ induced from $\mathcal{R}_1 \cong \mathbf{45} \oplus \overline{\mathbf{45}}$ with irreps $\mathbf{45}$, $\overline{\mathbf{45}}$ of M_{24} .

THE END

THANK YOU
FOR YOUR ATTENTION!